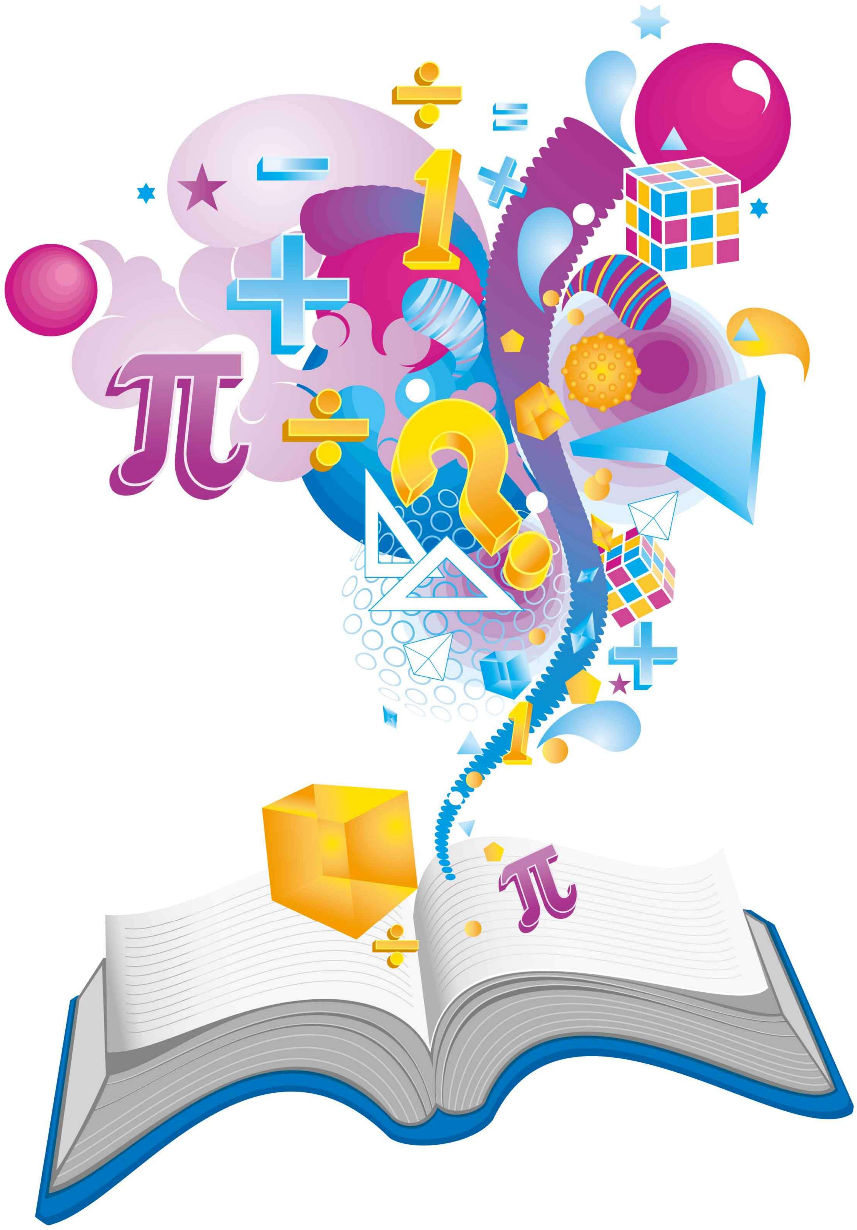
A-Level Mathematics



Bridging Course – Week 3

Coordinate Geometry

During the third week of this bridging course you will be reviewing your understanding of coordinate geometry and application to a range of questions.

You may wish to approach each section in one of two ways:

1. Systematically work through the topics;

- Read the notes and examples pages and watch video tutorials of any areas you need further instruction on, making your own notes on lined paper to file and keep.
- Complete as many of the questions in exercises 1 and 2 of the additional document as you need to feel confident with the concepts.
- Complete and mark the review test at the end of each section to assess your understanding.
- Repeat any sections as needed.

2. Assess before beginning;

- Complete and mark the end of section test.
- Review your work and identify areas of weakness.
- For these areas: watch the video tutorials and read the notes and examples pages, making your own notes on lined paper to file and keep.
- Complete as many of the questions in exercises 1 and 2 of the additional document as you need to help secure your understanding of that concept.
- Re-try the questions previously answered incorrectly from the end of section test.

For each section we have included "challenge" exercises, which you can attempt if you are confident with a topic and want to test your ability to apply skills to more complex problems.

Section A:

Points and Lines

1. Notes and examples on the following 7 pages of this document.

2. Helpful video tutorials for this topic:

Hegarty Maths Video Numbers Straight Lines: 210, 213 Parallel and Perpendicular: 214, 216
Corbett Maths Video Numbers Straight Lines: 194, 195 Parallel and Perpendicular: 196, 197
TL Maths https://sites.google.com/site/tlmaths314/home/gcse-to-a-level-maths-bridging-the-gap This link has all bridging the gap videos on – for this section: • 1 video on straight lines towards the bottom (after solving equations)

3. Pages 1 - 3 of the additional Coordinate Geometry booklet:
    - Exercises 1 and 2 provide opportunities to practise these skills.
    - Exercise 3 is a challenge exercise.
    - Answers for review are at the back of that booklet.

4. Additional sources of support:
    - CGP guide – "Head Start to A Level Maths"

5. Section test on pages **11-12** of this booklet, after the notes and examples (worked solutions for review at the end of booklet).

Section 1: Points and straight lines

Notes and Examples

These notes contain sub-sections on:

- Gradients, distances and mid-points
- The equation of a straight line
- The intersection of two lines

Gradients, distances and mid-points

The ***Explore resource Points*** looks at how you can find the midpoint of two points, the distance between two points and the gradient of a line joining two points.

You will have met gradients before at GCSE. Remember that lines which go “downhill” have negative gradients.

To find the gradient of a straight line between two points (x_1, y_1) and (x_2, y_2), use the formula

$$\text{gradient } = \frac{y_2 - y_1}{x_2 - x_1}.$$

If two lines are parallel, they have the same gradient.
If two lines with gradients m_1 and m_2 are perpendicular, then $m_1 m_2 = -1$

Example 1
P is the point (-3, 7). Q is the point (5, 1).
Calculate
(i) the gradient of PQ
(ii) the gradient of a line parallel to PQ
(iii) the gradient of a line perpendicular to PQ.

Solution
(i) Choose P as (x_1, y_1) and Q as (x_2, y_2).

or vice versa: it will still give the same answer (***WHY***?)

$$\text{Gradient of PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-7}{5-(-3)} = \frac{-6}{8} = -\frac{3}{4}$$

Notes:
(1) Draw a sketch and check that your answer is sensible (e.g. has negative gradient).
(2) Check that you get the same result when you choose Q as (x_1, y_1) and P as (x_2, y_2).

(ii) When two lines are parallel their gradients are equal. ($m_1 = m_2$)

So the gradient of the line parallel to PQ is also $-\frac{3}{4}$.

(iii) When two lines are perpendicular $m_1 m_2 = -1$.

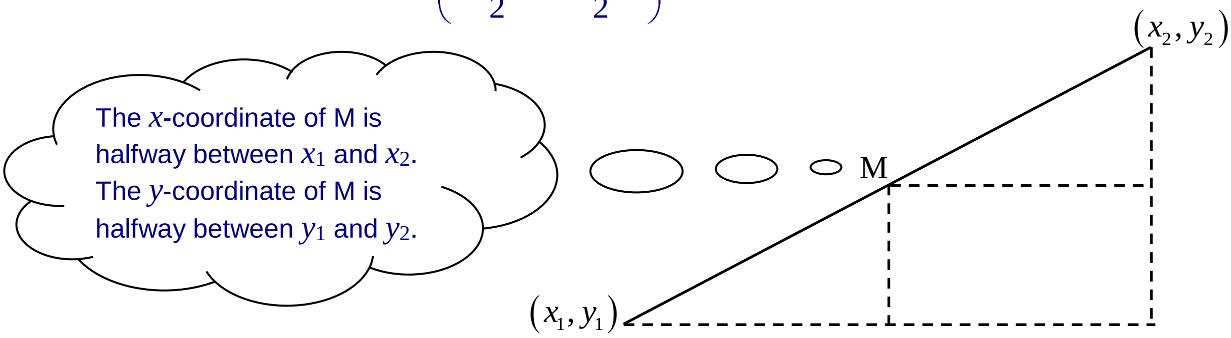
So $-\frac{3}{4}m_2 = -1$

$\Rightarrow m_2 = \frac{4}{3}$

The gradient of a line perpendicular to PQ is $\frac{4}{3}$.

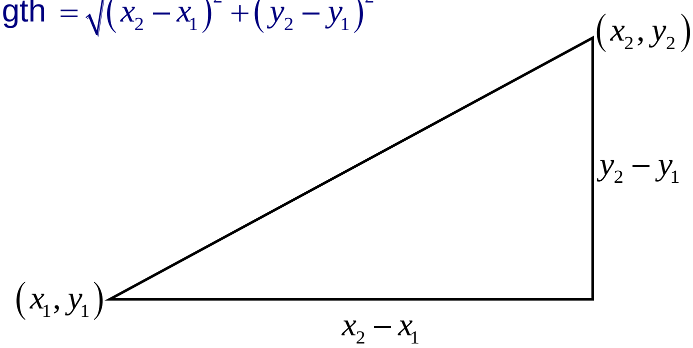
The midpoint of a line joining two points (x_1, y_1) and (x_2, y_2) is given by

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

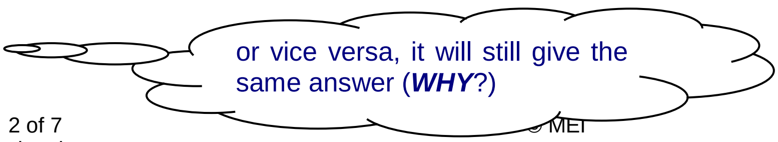


The length of a line joining two points (x_1, y_1) and (x_2, y_2) can be found using Pythagoras' Theorem.

$$\text{Length} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Example 2
A is the point (2, -6). B is the point (-3, 4).
Calculate
(i) the midpoint of AB
(ii) the length of AB.
Choose A as (x_1, y_1) and B as (x_2, y_2).



Solution

(i) Midpoint is $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$

i.e. $\left(\dfrac{2+(-3)}{2}, \dfrac{-6+4}{2}\right)$

$= \left(\dfrac{-1}{2}, -1\right)$

(ii) The distance AB is given by

$$\begin{aligned} d &= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \\ &= \sqrt{(2-(-3))^2+((-6)-4)^2} \\ &= \sqrt{(5)^2+(-10)^2} \\ &= \sqrt{25+100} \\ &= \sqrt{125} \end{aligned}$$

Note: The answer is often left like this if the square root is not exact. However since 125 = 25×5 then $\sqrt{125} = \sqrt{25}\sqrt{5} = 5\sqrt{5}$ is perhaps a simpler form.

For further practice in examples like the one above, try the ***Points skill pack***.

The equation of a straight line

The equation of a straight line is often written in the form $y = mx + c$, where m is the gradient and c is the intercept with the y-axis.

Example 3

Find (i) the gradient and (ii) the y-intercept of the following straight-line equations.

(a) $5y = 7x - 3$ (b) $3x + 8y - 7 = 0$

Solution

(a) Rearrange the equation into the form $y = mx + c$.

$5y = 7x - 3$ becomes $y = \frac{7}{5}x - \frac{3}{5}$

so $m = \frac{7}{5}$ and $c = -\frac{3}{5}$

Note the minus sign

(i) The gradient is $\frac{7}{5}$

(ii) The y-intercept is $-\frac{3}{5}$.

(b) Rearrange the equation into the form $y = mx + c$.

$3x + 8y - 7 = 0$ becomes $8y = -3x + 7$

giving $y = -\frac{3}{8}x + \frac{7}{8}$

so $m = -\frac{3}{8}$ and $c = \frac{7}{8}$

Note the minus sign

(i) The gradient is $-\frac{3}{8}$

(ii) The y-intercept is $\frac{7}{8}$.

Sometimes you may need to sketch the graph of a line. A sketch is a simple diagram showing the line in relation to the origin. It should also show the coordinates of the points where it cuts one or both axes.

You can explore straight line graphs using the Explore resources ***Straight lines*** and ***Parallel and perpendicular lines***. You may also find the Mathcentre video ***Equations of a straight line*** and ***Linear functions and graphs*** useful.

Example 4

Sketch the lines (a) $5y = 7x - 3$ (b) $3x + 8y - 7 = 0$

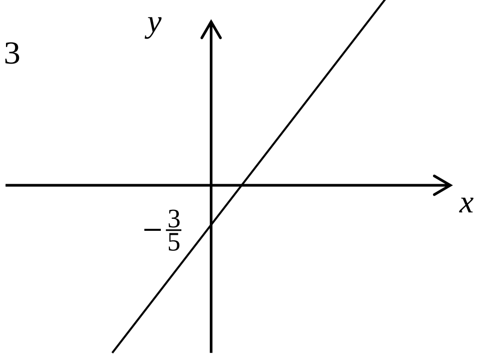
Solution

(a) From Example 3 you know that $5y = 7x - 3$ has gradient $\frac{7}{5}$ and y-intercept $-\frac{3}{5}$.

The gradient is positive, so the line slopes upwards from left to right. It's also greater than 1 and so steeper than 45 degrees.

This means the line goes through (0, -0.6) which is below the origin.

Sketch of $5y = 7x - 3$

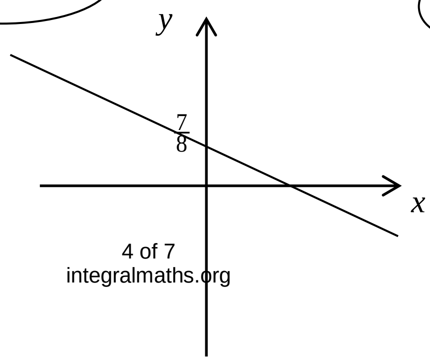


(b) From Example 3 you know that $3x + 8y - 7 = 0$ has gradient $-\frac{3}{8}$ and y-intercept $\frac{7}{8}$.

The gradient is negative, so the line slopes downwards from left to right. It is also and less than 1 and so less steep than 45 degrees.

This means the line goes through (0, 0.875) which is above the origin.

Sketch of $3x + 8y - 7 = 0$



OCR AS Maths Coordinate geometry 1 Notes & Examples

Sometimes you may need to find the equation of a line given certain information about it. If you are given the gradient and intercept, this is easy: you can simply use the form $y = mx + c$. However, more often you will be given the information in a different form, such as the gradient of the line and the coordinates of one point on the line (as in Example 5) or just the coordinates of two points on the line (as in Example 6).

In such cases you can use the alternative form of the equation of a straight line. For a line with gradient m passing through the point (x_1, y_1), the equation of the line is given by

$$y - y_1 = m(x - x_1).$$

Example 5

(i) Find the equation of the line with gradient 2 and passing through (3, -1).

(ii) Find the equation of the line perpendicular to the line in (i) and passing through (3, -1).

Solution

(i) The equation of the line is $y - y_1 = m(x - x_1)$

$m = 2$ and (x_1, y_1) is (3, -1)

$$\Rightarrow y - (-1) = 2(x - 3)$$
$$\Rightarrow y + 1 = 2x - 6$$
$$\Rightarrow y = 2x - 7$$

You should check that the point (3, -1) satisfies your line. If it doesn't, you must have made a mistake!

(ii) For two perpendicular lines $m_1 m_2 = -1$, so the gradient of the new line is $-\frac{1}{2}$.

The equation of the line is $y - y_1 = m(x - x_1)$

$m = -\frac{1}{2}$ and (x_1, y_1) is (3, -1)

$$\Rightarrow y - (-1) = -\tfrac{1}{2}(x - 3)$$
$$\Rightarrow -2y - 2 = x - 3$$
$$\Rightarrow -2y = x - 1$$
$$\Rightarrow y = -\tfrac{1}{2}x + \tfrac{1}{2}$$

The final form of the equation can be written in various different ways:
e.g. $2y = -x + 1$ (This form has no fractions.)
e.g. $2y + x = 1$ (This has no fractions and avoids having a negative sign at the start of the right hand side.)

OCR AS Maths Coordinate geometry 1 Notes & Examples

You can practice finding equations of lines using the ***Straight lines skill pack***. Also look at the ***Parallel and perpendicular lines skill pack***.

In the next example, you are given the coordinates of two points on the line.

Example 6
P is the point (3, 8). Q is the point (-1, 5).
Find the equation of PQ.

Solution
Choose P as (x_1, y_1) and Q as (x_2, y_2).

One method is to find the gradient and then use this value and one of the points in $y - y_1 = m(x - x_1)$

$$\text{Gradient of PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-8}{-1-3} = \frac{-3}{-4} = \frac{3}{4}$$

Now use

$$\begin{aligned} & y - y_1 = m(x - x_1) \\ \Rightarrow\ & y - 8 = \tfrac{3}{4}(x-3) \\ \Rightarrow\ & 4(y-8) = 3(x-3) \\ \Rightarrow\ & 4y - 32 = 3x - 9 \\ \Rightarrow\ & 4y = 3x + 23 \end{aligned}$$

You should check that P and Q satisfy your line.

An alternative approach to the above examples is to put the formula for m into the straight line equation to obtain

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

and then make the substitutions. This is equivalent to the first method, but does not involve calculating m separately first.

The intersection of two lines

The point of intersection of two lines is found by solving the equations of the lines simultaneously. This can be done in a variety of ways. When both equations are given in the form $y = \ldots$ then equating the right hand sides is a good approach (see below). If both equations are not in this form, you can re-arrange them into this form first, then apply the same method. Alternatively, you can use the elimination method if the equations are in an appropriate form.

Example 7

Find the point of intersection of the lines $y = 3x - 2$ and $y = 5x - 8$.

Solution

$3x - 2 = 5x - 8$

$\Rightarrow -2 = 2x - 8$

$\Rightarrow 6 = 2x$

$\Rightarrow x = 3$

Substitute into one of the equations to find y

Substituting x = 3 into $y = 3x - 2$ gives $y = 3 \times 3 - 2 = 7$

The point of intersection is (3, 7)

Check that (3, 7) satisfies the second equation.

Section 1: Points and straight lines

Section test

1. Which of the following points does **not** lie on the line $2y + 5x - 4 = 0$?

(a) (0.8, 0)
(b) (1, -0.5)
(c) (0, 2)
(d) (2, 3)

2. Here are four straight-line equations.

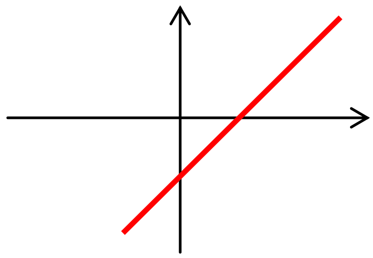
A $3y = 4x + 5$ B $4y = 3x - 1$
C $4y + 3x = 7$ D $4x + 3y = 2$

Which of the following statements are true? Choose as many as apply.

(a) Lines A and B are perpendicular
(b) Lines A and D are parallel
(c) Lines B and D are perpendicular
(d) Lines B and C are parallel
(e) Lines A and C are perpendicular

3. A straight line has equation $10y = 3x + 15$.
What is the gradient of the line?
What is the intercept of the line with the y-axis?

4. The diagram below shows the sketch of a straight line graph.



Which one of the equations below is a correct equation for this line?

(a) $y - x + 1 = 0$
(b) $y + x = 1$
(c) $y = x + 1$
(d) $y + x + 1 = 0$
(e) I don't know

5. P is the point (4, -2). Q is the point (-3, -5). What is the length PQ?

(a) $\sqrt{50}$
(b) $\sqrt{98}$
(c) $\sqrt{40}$
(d) $\sqrt{58}$

6. P is the point (3, 5). Q is the point (-1, 9).
What is the midpoint of PQ?

7. A straight line has a gradient of –2 and passes through the point (4, 1). What is its equation?

(a) $y + 2x = 6$ (b) $y = 2x - 6$
(c) $y + 2x - 9 = 0$ (d) $2y = x - 2$

8. The points A, B and C are (3, -2), (-1, 4) and (2, 3) respectively.
What is the equation of the line perpendicular to AB which passes through C?
Give your answer in the form $y = mx + c$.

9. The lines $y = 5x - 3$ and $y = 2x + 9$ intersect at P. What are the coordinates of P?

10. A is the point (1, 5), B is the point (4, 7) and C is the point (5, 2).
Triangle ABC is

(a) right-angled (b) scalene with no right angle
(c) equilateral (d) isosceles

Section B:

Circles

1. Notes and examples on the following 8 pages of this document.

2. Helpful video tutorials for this topic:

Hegarty Maths Video Numbers Circles, centre origin: 778, 779 Circles, centre not origin: 314-317
Corbett Maths Video Numbers Equation of Circle: 12

3. Pages 4 - 7 of the additional Coordinate Geometry booklet:
    a. Exercises 1 and 2 provide opportunities to practise these skills.
    b. Exercise 3 is a challenge exercise.
    c. Answers for review are at the back of that booklet.

4. Additional sources of support:
    - CGP guide – "Head Start to A Level Maths"

5. Section test on page **22** of this booklet, after the notes and examples (worked solutions for review at the end of booklet).

Section 2: Circles

Notes and Examples

These notes and examples contain subsections on

The equation of a circle

Start this section by looking at the Geogebra resource ***Circles***. First, set the centre of the circle to be the origin and vary the radius. Look at how the equation of the circle changes.

Now vary the coordinates of the centre of the circle, and look at how the equation of the circle changes.

You can also explore equations of circles using the ***Circles walkthrough***.

You should find out the following results, which you need to know:

The general equation of a circle, centre the origin and radius r is

$$x^2 + y^2 = r^2$$

The general equation of a circle, centre (a, b) and radius r is

$$(x-a)^2 + (y-b)^2 = r^2$$

Make sure you understand why these equations describe circles.

Example 1

For each of the following circles find (i) the coordinates of the centre and (ii) the radius.

(a) $x^2 + y^2 = 49$

(b) $(x+2)^2 + (y-6)^2 = 9$

Solution

(a) $x^2 + y^2 = 49$ can be written as $x^2 + y^2 = 7^2$.

(i) The coordinates of the centre are (0, 0)

(ii) The radius is 7.

This is a particular case of the general form $x^2 + y^2 = r^2$ which has centre (0, 0) and radius r.

(b) $(x+2)^2+(y-6)^2=9$ can be written as $(x-(-2))^2+(y-6)^2=3^2$.

(i) The coordinates of the centre are (–2, 6)

(ii) The radius is 3.

This is a particular case of the general form $(x-a)^2+(y-b)^2=r^2$ which has centre (a, b) and radius r.

Sometimes the circle equation needs to be rearranged into its standard form before you can find the centre and radius.

Example 2

Show that the equation $x^2+y^2+4x-6y-3=0$ represents a circle, and find its centre and radius.

Solution

The general equation of a circle is $(x-a)^2+(y-b)^2=r^2$

Multiplying out:

$$x^2-2ax+a^2+y^2-2by+b^2=r^2$$

$$x^2+y^2-2ax-2by+a^2+b^2-r^2=0$$

Comparing with the original equation:

$$-2a=4 \Rightarrow a=-2$$

$$-2b=-6 \Rightarrow b=3$$

$$a^2+b^2-r^2=-3 \Rightarrow 4+9-r^2=-3$$

$$\Rightarrow r^2=16$$

The equation can be written as $(x+2)^2+(y-3)^2=4^2$

This is the equation of a circle, centre (-2, 3), radius 4.

For practice in examples like the one above, try the ***Circle equations skill pack***.

Finding the equation of a circle

In the previous section you looked at different ways of finding the equation of a line. You can find the equation of a line from the gradient and the intercept, or from the gradient and one point on the line, or from two points on the line.

In the same way, there are several ways of finding the equation of a circle, depending on the information available.

OCR AS Maths Coordinate geometry 2 Notes & Examples

Finding the equation of a circle from the radius and centre

Example 3
Find the equation of each of the following.
(a) a circle, centre (0, 0) and radius 4.
(b) a circle, centre (3, –4) and radius 6.

Solution
(a) The equation of a circle centre the origin is $x^2 + y^2 = r^2$

$r = 4$ so the equation is $\quad x^2 + y^2 = 4^2$
i.e. $\quad x^2 + y^2 = 16$

(b) The equation of a circle centre (a, b) and radius r is $(x-a)^2 + (y-b)^2 = r^2$

$a = 3$, $b = -4$ and $r = 6$ so the equation is $\quad (x-3)^2 + (y-(-4))^2 = 6^2$
i.e. $\quad (x-3)^2 + (y+4)^2 = 36$

Finding the equation of a circle from its centre and one point on its circumference

If you know the centre of the circle and one point on its circumference, you can find the radius by calculating the distance between these two points. You can then find the equation of the circle.

Example 4
Find the equation of the circle, centre (1, -2), which passes through the point (-2, -3).

Solution
The distance r between (1, -2) and (-2, -3) is given by:

$$\begin{aligned} r &= \sqrt{(1-(-2))^2 + (-2-(-3))^2} \\ &= \sqrt{3^2 + 1^2} \\ &= \sqrt{10} \end{aligned}$$

The radius of the circle is therefore $\sqrt{10}$.
The equation of the circle is $(x-1)^2 + (y+2)^2 = 10$

Finding the equation of a circle from three points on its circumference

To find the equation of a line, you need the coordinates of two points on the line. To find the equation of a circle, you need the coordinates of three points on the circumference of the circle.

OCR AS Maths Coordinate geometry 2 Notes & Examples

The centre of the circle is the same distance from any point on the circumference. If the points A and B are on the circumference of a circle, the perpendicular bisector of A and B gives all the points that are the same distance from A and B. So the centre of the circle must be on the perpendicular bisector of A and B. Similarly it must be on the perpendicular bisectors of A and C, and B and C.

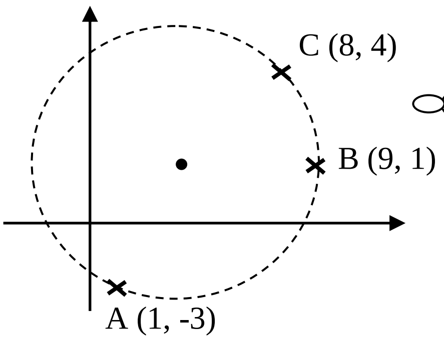
To find the centre of a circle through three points A, B and C, it is sufficient to find two of the perpendicular bisectors. For example, you can find the equations of the perpendicular bisectors of AB and BC, and then solve these equations simultaneously to find the point of intersection, i.e. the centre of the circle.

You can then use the coordinates of the centre and one of the three points A, B and C to find the radius of the circle (as in Example 4), and hence find the equation of the circle.

Example 5

Find the equation of the circle passing through A (1, –3), B (9, 1) and C (8, 4).

Solution



A sketch is often helpful. The sketch does not need to be accurate. It gives some idea of roughly where the centre is, so you can check your answer is reasonable.

You want to find the equation of the perpendicular bisector of AB. This is perpendicular to AB and passes through the midpoint M of AB.

The gradient of AB is found by using $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$m = \frac{1-(-3)}{9-1} = \frac{4}{8} = \frac{1}{2}$$

Note: Looking at the sketch we expect the gradient of AB to be positive.

Using $m_1 m_2 = -1$, the gradient of the perpendicular bisector is –2.

The midpoint M of AB is found by using M = $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

You are given A(1, -3) and B(9, 1) so M is $\left(\dfrac{1+9}{2}, \dfrac{-3+1}{2}\right)$ = (5, –1)

OCR AS Maths Coordinate geometry 2 Notes & Examples

The perpendicular bisector is found using $y - y_1 = m(x - x_1)$ with (x_1, y_1) = (5, 1) and $m = -2$.

so
$$y - (-1) = -2(x - 5)$$
$$y + 1 = -2x + 10$$
$$y = -2x + 9 \quad \text{(equation I)}$$

Next, use the same method to find the perpendicular bisector of BC.

The gradient of BC is $\frac{4-1}{8-9} = -3$

Note: Looking at the sketch, we expect the gradient of BC to be negative.

Therefore the gradient of the perpendicular bisector of BC is $\frac{1}{3}$.

The midpoint N of BC is $\left(\frac{9+8}{2}, \frac{1+4}{2}\right)$ so N is (8.5, 2.5).

The equation of the perpendicular bisector is
$$y - 2.5 = \tfrac{1}{3}(x - 8.5)$$
$$3(y - 2.5) = x - 8.5$$
$$3y - 7.5 = x - 8.5$$
$$3y = x - 1 \quad \text{(equation II)}$$

$$y = -2x + 9 \quad \text{(equation I)}$$
$$3y = x - 1 \quad \text{(equation II)}$$

Next, find the coordinates of the centre of the circle by solving equations (I) and (II) simultaneously.

Substituting (I) into (II)
$$3(-2x + 9) = x - 1$$
$$-6x + 27 = x - 1$$
$$28 = 7x$$
$$x = 4$$

Substituting $x = 4$ into equation (I) gives $y = -2(4) + 9 = 1$
So the coordinates of the centre are (4, 1).

Note: Looking at the sketch this appears to be a plausible result.

The radius is the distance between the centre (4, 1) and a point on the circumference such as (9, 1). This can be found by using $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

$$\text{radius} = \sqrt{(9-4)^2 + (1-1)^2} = \sqrt{25} = 5$$

Finally, using the general form $(x-a)^2+(y-b)^2=r^2$ with a = 4, b = 1 and r = 5 the equation of the circle is

$$(x-4)^2+(y-1)^2=25$$

Note: You should check that each of the points A, B and C satisfy this equation.

Circle geometry

The three facts about circles given below are important. They often help to solve problems involving circles.

1. The angle in a semicircle is a right angle.
2. The perpendicular from the centre of a circle to a chord bisects the chord.
3. The tangent to a circle is perpendicular to the radius at that point

You can see demonstrations of these properties using the ***Explore: Circle properties*** resource.

Keep these properties in mind when dealing with problems involving circles.

For some practice in using the third property, try the ***Tangent to a circle skill pack***.

The intersection of a line and a curve

Just as the point of intersection of two straight lines can be found by solving the equations of the two lines simultaneously, the point(s) of intersection of a line and a curve can be found by solving their equations simultaneously.

In many cases, the equations of both the line and the curve are given as an expression for y in terms of x. When this is the case, a sensible first step is to equate the expressions for y, as this leads to an equation in x only.

Example 6
Find the coordinates of the points where the line $y=x+2$ meets the curve $y=x^2-3x+5$.

Solution

$$x^2-3x+5=x+2$$
$$x^2-4x+3=0$$
$$(x-3)(x-1)=0$$
$$x=3 \text{ or } x=1$$

Equate the expressions for y to give an equation in x only

When $x = 3$ then $y = 3 + 2 = 5$
When $x = 1$ then $y = 1 + 2 = 3$.

Substitute the x values into the equation of the line

The points where the line meets the curve are (3, 5) and (1, 3).

You should check that each of these points satisfies the equation of the curve. (You have already used the equation of the line to find the y-values).

Notice that this problem involved solving a quadratic equation, which in this case had two solutions, showing that the line crossed the curve twice. However, the quadratic equation could have had no solutions, which would indicate that the line did not meet the curve at all, or one repeated solution, which would indicate that the line touches the curve.

The next example looks at the intersection of a line and a circle.

Example 7

Find the coordinates of the point(s) where the circle $(x+2)^2+(y-1)^2=9$ meets

(i) the line $y = 5$
(ii) the line $x = 1$
(iii) the line $y = 2 - x$

Solution

(i) Substituting $y = 5$ into the equation of the circle:

$$(x+2)^2+(5-1)^2=9$$
$$(x+2)^2+16=9$$
$$(x+2)^2=-7$$

The expression $(x + 2)^2$ cannot be negative

There are no solutions. The line does not meet the circle.

(ii) Substituting $x = 1$ into the equation of the circle:

$$(1+2)^2+(y-1)^2=9$$
$$9+(y-1)^2=9$$
$$(y-1)^2=0$$
$$y=1$$

The point is on the line $x = 1$, so its x-coordinate must be 1.

The line touches the circle at (1, 1).

(iii) Substituting $y = 2 - x$ into the equation of the circle:

$$(x+2)^2+(2-x-1)^2=9$$
$$(x+2)^2+(1-x)^2=9$$
$$x^2+4x+4+1-2x+x^2=9$$
$$2x^2+2x-4=0$$
$$x^2+x-2=0$$
$$(x-1)(x+2)=0$$
$$x=1 \text{ or } x=-2$$

Substitute the x values into the equation of the line to find the y-coordinates.

When $x = 1$, $y = 2 - 1 = 1$
When $x = -2$, $y = 2 - (-2) = 4$

The line crosses the circle at (1, 1) and (–2, 4).

For practice in examples like the one above, try the ***Circle and line intersection skill pack***.

The intersection of two curves

As before, the intersections of two curves can be found by solving the equations of the curves simultaneously. In many cases a sensible first step is to equate the expressions for y.

Example 8

Find the coordinates of the points where the curve $y=x^2-6x+5$ intersects the curve $y=-2x^2+12x-19$

Solution

Equate the expressions for y to give an equation in x only

$$x^2-6x+5=-2x^2+12x-19$$
$$3x^2-18x+24=0$$

There is a common factor of 3, so divide by 3 before factorising.

$$x^2-6x+8=0$$
$$(x-2)(x-4)=0$$
$$x=2 \text{ or } x=4$$

When $x = 2$, $y = (2)^2 - 6(2) + 5 = -3$
When $x = 4$, $y = (4)^2 - 6(4) + 5 = -3$

Substitute the x values into the equation of one of the curves (in this case the first one).

The points of intersection are (2, –3) and (4, –3).

You should check that each of these points satisfy the equation of the second curve. (You have already used the equation of the first curve to find the y-values).

Section 2: Circles

Section test

1. A circle has the equation $x^2 + y^2 = 16$.
   What is the radius of this circle?

2. A circle has the equation $(x+3)^2 + (y-1)^2 = 4$.
   Which of the following statements is false? Choose as many as apply.

(a) The y coordinate of the centre is –1
(b) The radius of the circle is 2
(c) The x coordinate of the centre is –3
(d) The point (–3,–1) lies on the circle

3. The equation of a circle with centre (2, 1) and radius 6 is

(a) $(x+2)^2 + (y+1)^2 = 36$
(b) $(x+2)^2 + (y+1)^2 = 6$
(c) $(x-2)^2 + (y-1)^2 = 6$
(d) $(x-2)^2 + (y-1)^2 = 36$

4. The equation of a circle with radius 5 and centre (3, -2) can be written as

(a) $x^2 + y^2 - 3x + 2y = 25$
(b) $x^2 + y^2 + 3x - 2y = 25$
(c) $x^2 + y^2 - 6x + 4y = 12$
(d) $x^2 + y^2 + 6x - 4y = 12$

5. A circle has equation $x^2 + y^2 - 2x + 6y = 10$. Find the centre and radius of the circle.

6. O is the centre of a circle. The point P (2, 4) lies on the circumference of the circle. What is the gradient of the tangent at P?

7. The equation of a line is $y = x$. The equation of a circle is $x^2 + y^2 = 8$.
   Which one of the following statements is true?

(a) The line does not meet the circle
(b) The line cuts the circle at two points
(c) The line touches the circle

8. AB is the diameter of a circle centre O. P is a point on the circumference.
   Which one of the following statements is true?

(a) When P is equidistant from A and B then OP is parallel to AB
(b) Angle APB varies as the position of P varies
(c) $AP^2 + PB^2 = AB^2$
(d) Triangle APB is acute angled

9. The line $y = 2x + 3$ is a tangent to a circle with centre (2, -3).
   The radius of the circle is

(a) $\sqrt{20}$
(b) $\sqrt{40}$
(c) 20
(d) 40

10. The line $y = 2x$ does not meet the circle $(x-2)^2 + (y-1)^2 = d$.
    Find the range of possible values for d.

SOLUTIONS TO SECTION REVIEW TESTS

OCR AS Maths Coordinate geometry 1 Section test

Solutions to section test

1. $2\times 0+5\times 0.8-4=0+4-4=0$
   $2\times -0.5+5\times 1-4=-1+5-4=0$
   $2\times 2+5\times 0-4=4+0-4=0$
   $2\times 3+5\times 2-4=6+10-4\neq 0$

2. Line A can be written as $y=\frac{4}{3}x+\frac{5}{3}$
   Line B can be written as $y=\frac{3}{4}x-\frac{1}{4}$
   Line C can be written as $y=-\frac{3}{4}x+\frac{7}{4}$
   Line D can be written as $y=-\frac{4}{3}x+\frac{2}{3}$

   None of the lines are parallel, since they all have different gradients.
   Lines A and B are not perpendicular, since $\frac{4}{3}\times\frac{3}{4}\neq -1$.

   Lines B and D are perpendicular, since $\frac{3}{4}\times -\frac{4}{3}=-1$
   Lines A and C are perpendicular, since $\frac{4}{3}\times -\frac{3}{4}=-1$

3. The equation of the line can be written as $y=0.3x+1.5$
   So the gradient is 0.3 and the y-intercept is 1.5.

4. $y-x+1=0$ can be written as $y=x-1$
   $y+x=1$ can be written as $y=-x+1$
   The third equation is $y=x+1$
   $y+x+1=0$ can be written as $y=-x-1$
   The line in the diagram has a positive gradient and a negative intercept, so $y-x+1=0$ is the correct equation.

5. $$\begin{aligned}\text{Length PQ} &= \sqrt{(4-(-3))^2+(-2-(-5))^2}\\ &= \sqrt{7^2+3^3}\\ &= \sqrt{49+9}\\ &= \sqrt{58}\end{aligned}$$

6. $$\text{Midpoint }=\left(\frac{3+(-1)}{2},\frac{5+9}{2}\right)=(1,7)$$

7. $$\begin{aligned}y-y_1&=m(x-x_1)\\ y-1&=-2(x-4)\\ y-1&=-2x+8\\ y+2x-9&=0\end{aligned}$$

8. Gradient of AB $= \frac{4-(-2)}{-1-3} = \frac{6}{-4} = -\frac{3}{2}$

Gradient of line perpendicular to AB $= \frac{2}{3}$

Equation of line with gradient $\frac{2}{3}$ passing through C (2, 3) is

$$y-3 = \tfrac{2}{3}(x-2)$$
$$y-3 = \tfrac{2}{3}x - \tfrac{4}{3}$$
$$y = \tfrac{2}{3}x + \tfrac{5}{3}$$

9. $5x - 3 = 2x + 9$

$3x = 12$

$x = 4$

When $x = 4$, $y = 5 \times 4 - 3 = 17$

The coordinates of point of intersection are (4, 17).

10. Length AB $= \sqrt{(4-1)^2 + (7-5)^2} = \sqrt{9+4} = \sqrt{13}$

Length BC $= \sqrt{(4-5)^2 + (7-2)^2} = \sqrt{1+25} = \sqrt{26}$

Length AC $= \sqrt{(5-1)^2 + (2-5)^2} = \sqrt{16+9} = \sqrt{25}$

The sides are all different lengths.

Gradient AB $= \frac{7-5}{4-1} = \frac{2}{3}$

Gradient BC $= \frac{7-2}{4-5} = \frac{5}{-1} = -5$

Gradient AC $= \frac{2-5}{5-1} = \frac{-3}{4} = -\frac{3}{4}$

None of the lines are perpendicular, so there is no right-angle.

The triangle is scalene with no right-angle.

OCR AS Maths Coordinate geometry 2 Section test solutions

Solutions to section test

1. $x^2 + y^2 = 16$

   Comparing with the standard equation $x^2 + y^2 = r^2$ for a circle with centre O and radius r, gives $r^2 = 16 \Rightarrow r = 4$.
   The radius of the circle is 4.

2. The equation $(x+3)^2 + (y-1)^2 = 4$ represents a circle with centre (-3, 1) and radius 2.
   So the statement is that the y coordinate of the centre is -1 is the only incorrect one.

3. The equation of a circle with centre (a, b) and radius r is
   $$(x-a)^2 + (y-b)^2 = r^2$$
   so the equation of a circle with centre (2, 1) and radius 6 is
   $$(x-2)^2 + (y-1)^2 = 6^2$$
   $$(x-2)^2 + (y-1)^2 = 36$$

4. The equation of a circle with centre (3, -2) and radius 5 is
   $$(x-3)^2 + (y+2)^2 = 5^2$$
   $$x^2 - 6x + 9 + y^2 + 4y + 4 = 25$$
   $$x^2 + y^2 - 6x + 4y = 12$$

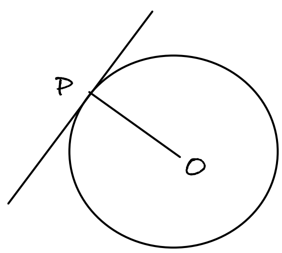
5. $x^2 + y^2 - 2x + 6y = 10$

   $x^2 - 2x + y^2 + 6y = 10$

   $(x-1)^2 - 1 + (y+3)^2 - 9 = 10$

   $(x-1)^2 + (y+3)^2 = 20$

   The centre of the circle is (1, -3).
   The radius is $\sqrt{20}$.



6. The tangent at P is perpendicular to the radius OP.
   The gradient of OP is 2
   so the gradient of the tangent is $-\frac{1}{2}$.

7. Substituting $y = x$ into $x^2 + y^2 = 8$
   gives $x^2 + x^2 = 8$

   $2x^2 = 8$

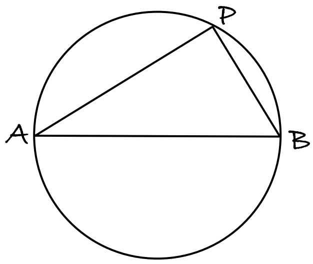
   $x^2 = 4$

   $x = \pm 2$

   There are two distinct solutions, so the line cuts the circle at two points.

OCR AS Maths Coordinate geometry 2 Section test solutions

8. Since AB is a diameter, the angle APB is the angle in a semicircle and so angle APB is 90°.



Since triangle APB is right-angled, Pythagoras' theorem applies, and therefore
$AP^2 + PB^2 = AB^2$

9. Equation of circle is $(x-2)^2 + (y+3)^2 = r^2$
Substituting in $y = 2x + 3$ gives

$$(x-2)^2 + (2x+3+3)^2 = r^2$$

$$(x-2)^2 + (2x+6)^2 = r^2$$

$$x^2 - 4x + 4 + 4x^2 + 24x + 36 = r^2$$

$$5x^2 + 20x + 40 - r^2 = 0$$

Since the line is a tangent, it touches the circle, so the quadratic equation has a repeated root and therefore the discriminant must be zero.
For the quadratic equation, $a = 5$, $b = 20$, $c = 40 - r^2$.
Discriminant $= b^2 - 4ac = 0$

$$20^2 - 4 \times 5(40 - r^2) = 0$$

$$400 = 20(40 - r^2)$$

$$20 = 40 - r^2$$

$$r^2 = 20$$

$$r = \sqrt{20}$$

10. $(x-2)^2 + (y-1)^2 = d$

$(x-2)^2 + (2x-1)^2 = d$

$x^2 - 4x + 4 + 4x^2 - 4x + 1 = d$

$5x^2 - 8x + 5 - d = 0$

Since the line does not meet the circle, discriminant of this equation < 0.

$a = 5, b = -8, c = 5 - d$

$(-8)^2 - 4 \times 5(5 - d) < 0$

$64 - 100 + 20d < 0$

$20d < 36$

$d < \frac{9}{5}$