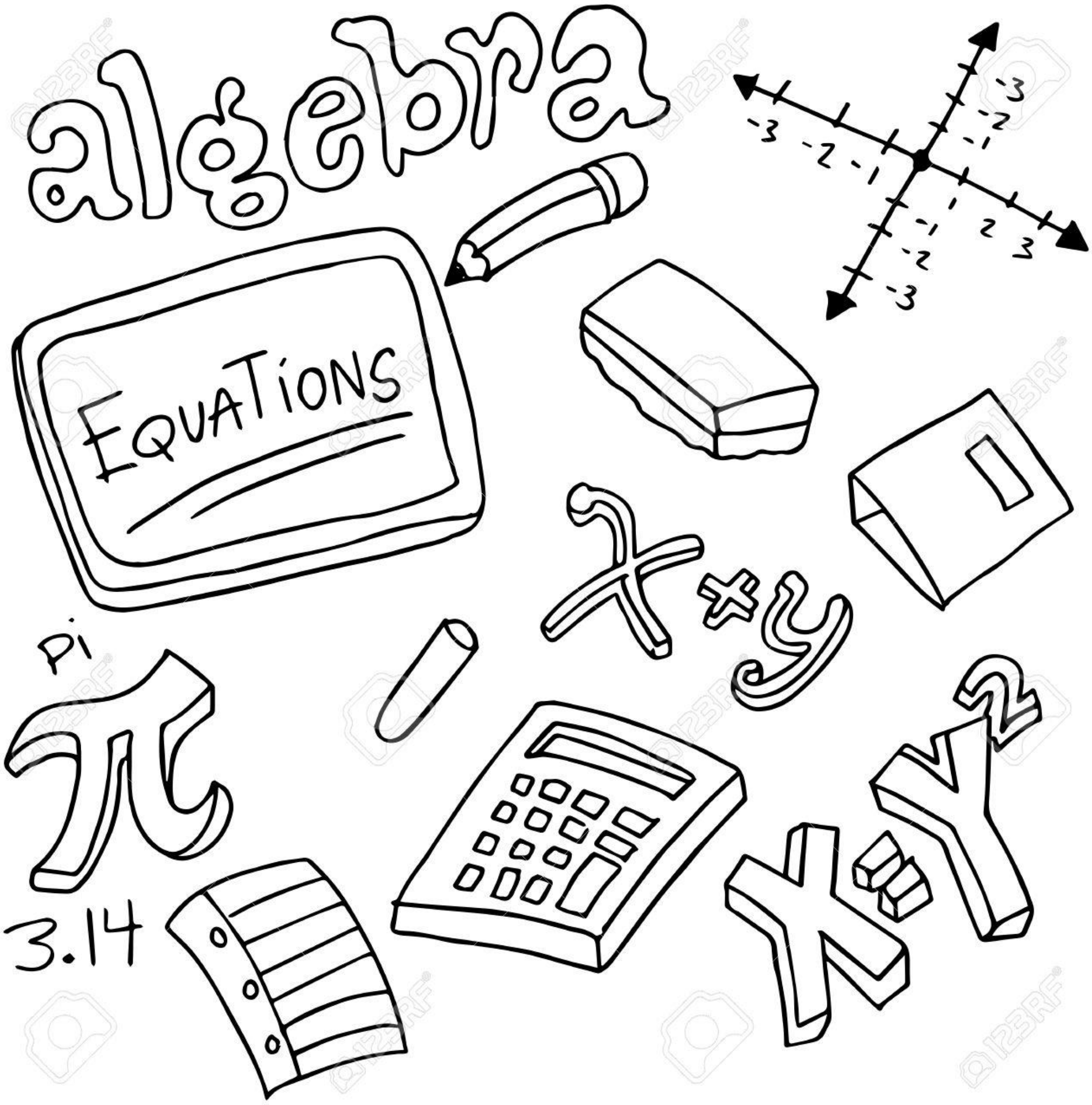
A-Level Mathematics



Bridging Course – Week 1

Entry Requirements for Studying A-level Mathematics?

- Students who are expected to achieve at least a grade 7 in GCSE Mathematics.
- Students who have enjoyed their GCSE Mathematics course, and who are keen problem solvers.
- Students who enjoy spending time working through a series of logical steps to find a conclusion.
- Students who enjoy the challenge of working through a difficult problem, and have the resilience required to keep going if they find something hard.
- Students who enjoy discussing their thoughts and ideas with others in a group.

What to expect from A-level Mathematics.

Just as languages provide the building blocks and rules we need to communicate, mathematics uses its own language, made up of numbers, symbols and formulae, to explore the rules we need to measure or identify essential problems like distance, speed, time, space, change, force and quantities. Studying maths helps us find patterns and structure in our lives. Practically, maths helps us put a price on things, create graphics, build websites, build skyscrapers and generally understand how things work or predict how they might change over time and under different conditions. As a subject, maths is continually growing and changing, as mathematician and scientists expand on what they already know to discover new theories and inventions.

A-level Mathematics is a demanding course. However, if you are a resilient and determined problem solver, it can also be extremely rewarding. Students should expect to complete regular pieces of written and online homework, as well as substantial amounts of independent practise, in order to fully embed concepts in the mind and apply them to a range of different situations. Communication of mathematics is as important as finding the correct answers and this will be tested through written work and regular assessments. Engagement with class discussion is essential in order to develop these communication skills.

Whilst A-Level Mathematics is a fascinating course to study in itself, it is also what is termed a "facilitating" subject; it will help you to study lots of other subjects and pursue lots of different careers. Maths helps support the study of subjects like physics, chemistry, engineering, IT, economics, business and biology. Even in essay-based subjects like History, maths can be useful as it teaches you to think in a logical way, something which is vital when putting across a coherent logical argument. Maths is one of the best subjects to develop your analytical, research and problem-solving skills. Not only will studying maths help give you the knowledge to tackle scientific, mechanical, coding and abstract problems, it will also help you develop logic to tackle everyday issues like planning projects, managing budgets and even debating effectively.

This bridging course will provide you with a mixture of information about A-level Mathematics, and what to expect from the course, as well as key work to complete. Students who are expecting to study Mathematics at A-level, and are likely to meet the entry requirements, must complete the bridging course fully and thoroughly, to the best of their ability. You should complete all work on paper and keep it in a file, in an ordered way. You will submit it to your teacher in September. All of the work will be reviewed and selected work will be assessed, and you will be given feedback on it. This work will be signalled to you. If you do not have access to the internet, please contact the school and appropriate resources will be sent to you.

If you are thinking about studying Mathematics at A-level you should attempt this work to see whether or not you think studying a subject like this is right for you. If you later decide to study Mathematics, you must ensure you complete this work in full. This work should be completed after you have read and completed the Study Skills work that all of Year 12 should complete.

Course outline

Paper 1 – Pure Mathematics worth $33^1/_3$%	Paper 2 – Pure and Statistics worth $33^1/_3$%	Paper 3 – Pure and Mechanics worth $33^1/_3$%
• A written exam lasting 2 hours. • A calculator is allowed. • Any aspect of pure mathematics could be tested in this paper.	• A written exam lasting 2 hours. • A calculator is allowed. • Section A - Any aspect of pure maths could be tested in this paper, even if already tested in paper 1. • Section B - Any statistical technique could be tested in the second half of the paper.	• A written exam lasting 2 hours. • A calculator is allowed. • Section A - Any aspect of pure maths could be tested in this paper, even if already tested in paper 1 or 2. • Section B - Any mechanics topics could be tested in the second half of the paper.

There is no controlled assessment (coursework) for A-Level Mathematics; your grade is based solely on the three exams detailed above.

Throughout year 12 and year 13, you will be assessed regularly through written homework tasks and in-class assessments. These will give you and your teachers opportunity to be continually reviewing your areas of strengths and weakness and give you a focus for your independent study.

At the end of year 12 and throughout year 13 you will be given exam papers to ensure you have plenty of practise answering exam questions and developing exam technique.

The following work requires a lot of time and careful practise in order to ensure techniques are properly embedded and ready to be developed and built upon. Remember to **take regular breaks**, complete as much or as little additional practise as you need to successfully complete the section test for each section and **try your best**. There are links to instructional videos included, as well as notes and examples to help you with each topic area. Your maths teacher will review each of these topics during year 12.

Algebraic Manipulation

During the first week of this bridging course you will be reviewing your understanding and ability to apply knowledge and skills of manipulating algebraic expressions and solving linear and quadratic equations and inequalities.

You may wish to approach each section in one of two ways:

1. Systematically work through the topics;

- Read the notes and examples pages and watch video tutorials of any areas you need further instruction on, making your own notes on lined paper to file and keep.
- Complete as many of the questions in exercises 1 and 2 of the additional document as you need to feel confident with the concepts.
- Complete and mark the review test at the end of each section to assess your understanding.
- Repeat any sections as needed.

2. Assess before beginning;

- Complete and mark the end of section test.
- Review your work and identify areas of weakness.
- For these areas: watch the video tutorials and read the notes and examples pages, making your own notes on lined paper to file and keep.
- Complete as many of the questions in exercises 1 and 2 of the additional document as you need to help secure your understanding of that concept.
- Re-try the questions previously answered incorrectly from the end of section test.

For each section we have included "challenge" exercises, which you can attempt if you are confident with a topic and want to test your ability to apply skills to more complex problems.

Section A:

Polynomials – graphs and manipulation

1. Notes and examples on the following 7 pages of this document.

2. Helpful video tutorials for this topic:

Hegarty Maths Video Numbers Basic Algebraic Manipulation: 157-159 Expanding brackets (single, double and triple): 161-166 Factorising: 167-171
Corbett Maths Video Numbers Basic Algebraic Manipulation: 9, 16, 18, 11 Expanding Brackets: 13, 14, 15 Factorising: 117
TL Maths https://sites.google.com/site/tlmaths314/home/gcse-to-a-level-maths-bridging-the-gap This link has all bridging the gap videos on – for this section: • Expanding brackets (x2) • Factorising (x3) • Rearranging Formulae (x2)

3. Pages 1 - 4 of the additional Algebra booklet:
    - Exercises 1 and 2 provide opportunities to practise these skills.
    - Exercise 3 is a challenge exercise.
    - Answers for review are at the back of that booklet.

4. Additional sources of support:
    - CGP guide – “Head Start to A Level Maths”

5. Section test on pages **<u>13-14</u>** of this booklet, after the notes and examples (worked solutions for review at the end of booklet).

Section 1: Polynomial functions and graphs

Notes and Examples

These notes contain subsections on

Adding and subtracting polynomials

Example 1

For the polynomials $\mathrm{f}(x) = 2x^3 - 3x^2 + 1$

$\mathrm{g}(x) = x^3 + x^2 - 3x - 4$

find (i) $\mathrm{f}(x) + \mathrm{g}(x)$

(ii) $\mathrm{f}(x) - \mathrm{g}(x)$

Solution

(i) $$\begin{aligned}\mathrm{f}(x) + \mathrm{g}(x) &= (2x^3 - 3x^2 + 1) + (x^3 + x^2 - 3x - 4)\\ &= (2x^3 + x^3) + (-3x^2 + x^2) + (-3x) + (1 - 4)\\ &= 3x^3 - 2x^2 - 3x - 3\end{aligned}$$

Add the terms in x^3, the terms in x^2, the terms in x and the constant terms separately. Notice that $\mathrm{f}(x)$ does not have a term in x. It is not essential to put in the brackets as shown here, but it can be helpful.

$$\begin{array}{rrrrr} & 2x^3 & -3x^2 & & +1 \\ + & x^3 & +x^2 & -3x & -4 \\ \hline & 3x^3 & -2x^2 & -3x & -3 \end{array}$$

Alternatively you can write it out like an addition sum:

(ii) $$\begin{aligned}\mathrm{f}(x) - \mathrm{g}(x) &= (2x^3 - 3x^2 + 1) - (x^3 + x^2 - 3x - 4)\\ &= (2x^3 - x^3) + (-3x^2 - x^2) + (-3x) + (1 + 4)\\ &= x^3 - 4x^2 - 3x + 5\end{aligned}$$

Do this in a similar way, but be careful about signs.

$$\begin{array}{rrrrr} & 2x^3 & -3x^2 & & +1 \\ - & x^3 & +x^2 & -3x & -4 \\ \hline & x^3 & -4x^2 & +3x & +5 \end{array}$$

Alternatively (again be careful about signs):

OCR AS Maths Polynomials 1 Notes and Examples

Multiplying polynomials

You are already familiar with multiplying out two linear expressions to obtain a quadratic expression.

You now need to be able to deal with more complicated multiplications. The principle is the same – for a multiplication involving two sets of brackets, each term in one set of brackets needs to be multiplied by each term in the other set of brackets.

First of all, it is helpful to think about how many terms there will be in the result (before simplifying). Each of the 3 terms in the first set of brackets must be multiplied by each of the terms in the second set of brackets, so there should be 9 terms altogether.

There are a number of different ways of setting out this multiplication. One way is shown in Example 2 below.

Each term in the first set of brackets is multiplied by the whole of the second set of brackets

Example 2
Multiply x^2+3x-2 by $2x^2-x+4$

Solution

9 terms as expected

$$\begin{aligned}(x^2+3x-2)(2x^2-x+4) &= x^2(2x^2-x+4)+3x(2x^2-x+4)-2(2x^2-x+4)\\ &= (2x^4-x^3+4x^2)+(6x^3-3x^2+12x)+(-4x^2+2x-8)\\ &= 2x^4+(-x^3+6x^3)+(4x^2-3x^2-4x^2)+(12x+2x)-8\\ &= 2x^4+5x^3-3x^2+14x-8\end{aligned}$$

Collect like terms together

You also need to be able to multiply out expressions involving more than two sets of brackets. This is shown in the next example.

Example 3
Multiply out $(x-2)(2x+3)(3x-1)$

Solution
It is often easiest to multiply out two sets of brackets, and then multiply the result by the third set of brackets.

$$\begin{aligned}(x-2)(2x+3) &= 2x^2+3x-4x-6\\ &= 2x^2-x-6\end{aligned}$$

Simplify here or the next step will be more complicated!

$$\begin{aligned}(x-2)(2x+3)(3x-1) &= (2x^2-x-6)(3x-1)\\ &= 2x^2(3x-1)-x(3x-1)-6(3x-1)\\ &= 6x^3-2x^2-3x^2+x-18x+6\\ &= 6x^3-5x^2-17x+6\end{aligned}$$

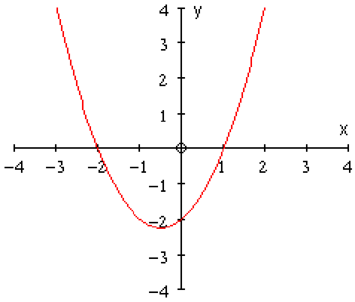
You can of course use another approach, such as using a table, or doing it in your head, to multiply the quadratic by the third set of brackets.

Graphs of polynomial functions

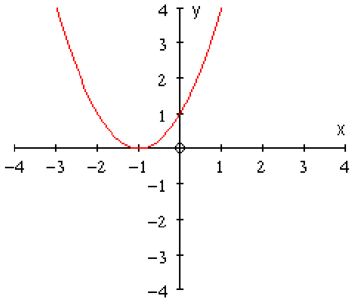
There are two simple rules about the graphs of polynomial functions:

- a polynomial of degree n meets the x-axis at most n times.
- a polynomial of degree n has at most $n - 1$ turning points.

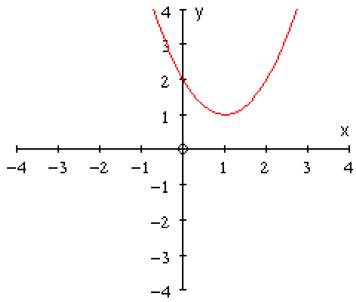
For example, a quadratic, which has degree 2, has 1 turning point and meets the x-axis at most twice.



This quadratic crosses the x-axis twice.

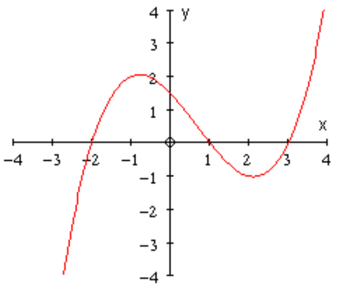


This quadratic touches the x-axis once.

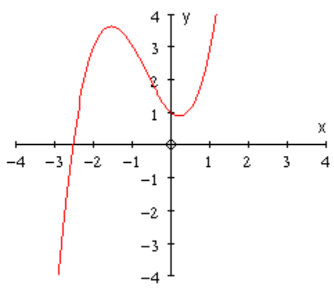


This quadratic does not meet the x-axis at all.

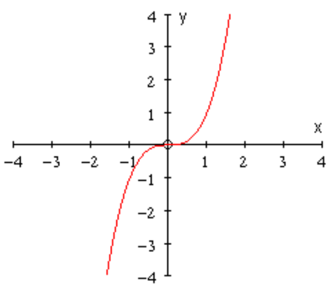
A cubic, which has degree 3, has at most 2 turning points and meets the x-axis at most three times and at least once. Here are some examples.



This cubic graph has two turning points and crosses the x-axis three times.



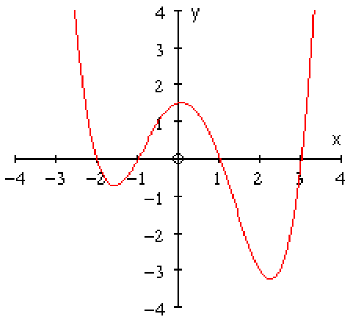
This cubic graph has two turning points, but only crosses the x-axis once.



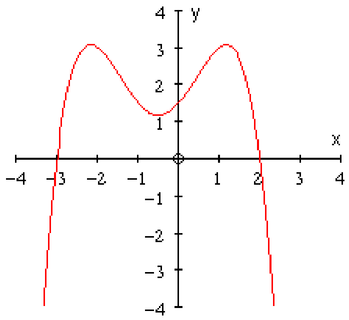
The simplest cubic graph of all, $y = x^3$, has just one turning point, a point of inflection. You can think of this as two turning points in the same place. It crosses the x-axis just once.

OCR AS Maths Polynomials 1 Notes and Examples

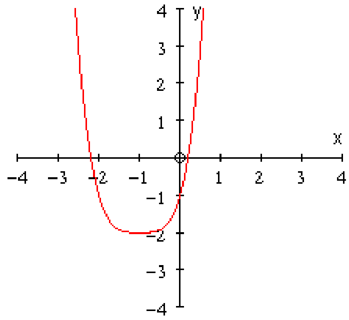
A quartic, which has degree 4, has at most 3 turning points and meets the x-axis at most four times. Here are some examples.



This quartic graph has three turning points and crosses the x-axis four times.



This quartic graph has three turning points and crosses the x-axis twice.



This quartic graph has just one turning point and crosses the x-axis twice. Notice that the turning point is much flatter than usual – it is actually three turning points all in the same place. This makes the shape different from a quadratic graph.

Sketching graphs of polynomials in factorised form

You have already done some work on factorising quadratic expressions. As you know, a quadratic expression can sometimes be factorised into two linear factors. These factors can be used to tell you where the graph of the quadratic cuts the x-axis.

For example, the quadratic expression $y = x^2 + x - 2$
can be written in factorised form as $y = (x-1)(x+2)$

When the graph cuts the x-axis, the value of $y = 0$.

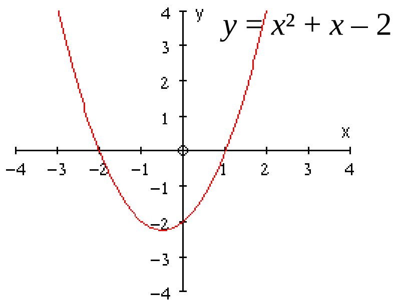
$$(x-1)(x+2) = 0$$
$$x = 1 \text{ or } x = -2$$

The graph therefore cuts the x-axis at (1, 0) and (-2, 0).

You can also find out where the graph cuts the y-axis by substituting $x = 0$. In this case, when $x = 0$, $y = -2$, so the graph cuts the y-axis at (0, -2).

This information allows you to sketch the graph.

OCR AS Maths Polynomials 1 Notes and Examples



It is also useful to think about the behaviour of the graph as $x \to \infty$ (i.e. when x is very large and positive) and as $x \to -\infty$ (i.e. when x is very large and negative). For the quadratic graph above, for values of x which are numerically large the term x^2 is the most significant term (the dominant term). So you just need to think about whether this term is positive or negative. In this case it is positive as $x \to \infty$ and as $x \to -\infty$. So the graph disappears off the top of the page at both the left and right of the graph.

The same ideas can be extended to any polynomial. You will learn how to factorise cubics and higher degree polynomials in Section 2. For now, you will look at polynomials that are given in factorised form.

Examination questions commonly ask you to sketch the graph of a function which you have already factorised.

This should be an easy source of marks (provided you have managed the factorising!) but a lot of candidates throw marks away. Here are some tips:

- A sketch means a sketch! Do it in the answer booklet, not on graph paper (if graph paper is provided that
- does not mean that you are expected to use it!). You should certainly not be calculating and plotting points – this is very time-consuming, unnecessary and may result in only part of the graph.

- Show the points at which the graph cuts BOTH axes. The points at which the graph cuts the x-axis can be found from the factorised equation. Find the point at which the graph cuts the y-axis by substituting $x = 0$, and mark on this point – one mark is often given for this.

- Make sure that your graph is the right way up. If you have marked on the point where the graph cuts the y-axis then it should be clear how the graph goes. Also think about whether y is positive or negative when x is very large and positive, and when x is very large and negative.

- Do not stop the graph when you reach an axis! You may lose marks for this. The graph should go beyond each of the points marked.

- If you have a graphical calculator, do use it to check, but don't rely on it. Sometimes the output can be misleading - the scale used may mean that not all the important features are visible. If the question requires you to calculate exact values of intersections, then you will need to show your working and not just read off the graph.

Example 4

Sketch the following graphs, showing the points where the graphs meet the coordinate axes.

(i) $y = (x-1)(x+3)(2x+1)$

(ii) $y = x(x-2)^2(x+2)$

Solution

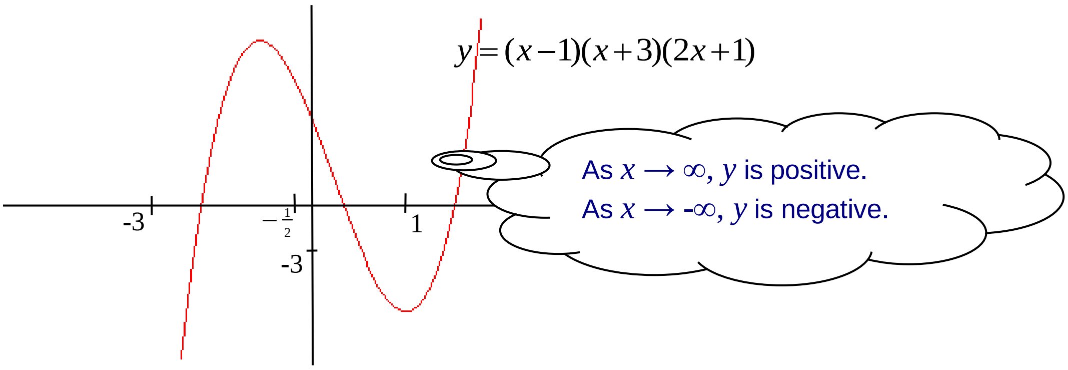
(i) $y = (x-1)(x+3)(2x+1)$

When $y = 0$, $x = 1$, $x = -3$ or $x = -\frac{1}{2}$.

When $x = 0$, $y = -1 \times 3 \times 1 = -3$

The graph crosses the x-axis at (1, 0), (-3, 0) and $(-\frac{1}{2}, 0)$.

The graph crosses the y-axis at (0, -3).



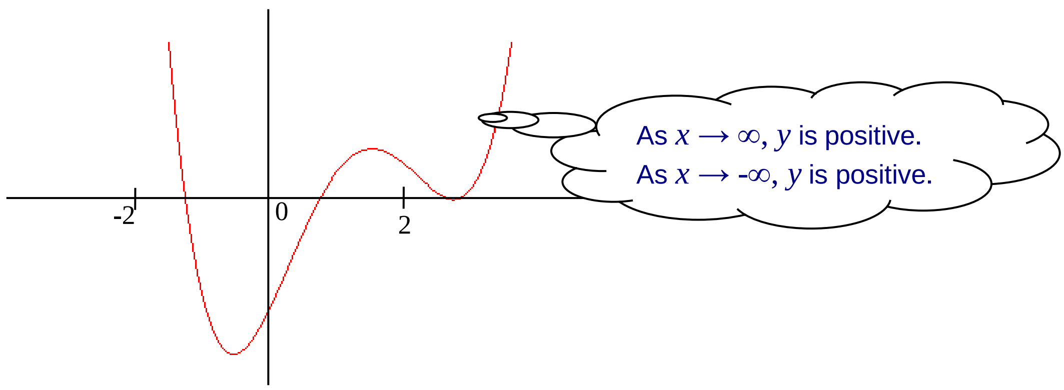
(ii) $y = x(x-2)^2(x+2)$

When $y = 0$, $x = 0$, $x = 2$ (repeated) or $x = -2$.

When a root is repeated, the graph touches the x-axis

There is no need to look at where the graph crosses the y-axis, since you already know it passes through the origin.

The graph crosses the x-axis at (0, 0) and (-2, 0), and touches the x-axis at (2, 0).



Finding the equation of a curve

If you are know where a polynomial curve cuts the axes (or alternatively, if you are given the roots of a polynomial equation), you can deduce the equation of the curve by writing it in factorised form and then multiplying out the brackets.

Example 5
Find a polynomial equation which has roots x = 1, x = -3 and x = 0.5.

Solution
A polynomial equation with these roots is $(x-1)(x+3)(2x-1)=0$.

Multiplying out: $$\begin{aligned}(x-1)(x+3)(2x-1) &= (x^2+2x-3)(2x-1)\\ &= 2x^3+4x^2-6x-x^2-2x+3\\ &= 2x^3+3x^2-8x+3\end{aligned}$$

An equation with these roots is $2x^3+3x^2-8x+3=0$

Note that any multiple of this polynomial (e.g. $4x^3+6x^2-16x+6=0$) would have the same roots. This is the simplest possible equation.

Section 1: Polynomial functions and graphs

Section test

Questions 1 – 5 are about the polynomials

$$f(x) = x^4 + 2x^3 + 2x - 1$$
$$g(x) = 3x^3 + 4x^2 - 2x + 5$$

1. Find f(x) + g(x).

2. Find f(x) – g(x).

3. Find $(x-2)f(x)$.

4. Find $(2x+1)g(x)$.

5. What is the degree of the polynomial obtained by multiplying f(x) by g(x)?

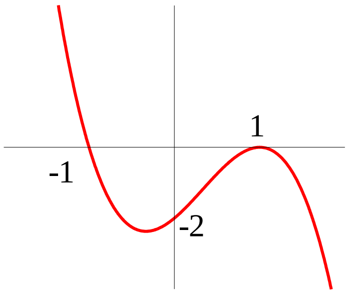
6. When the expression $(x^2 - x + 1)(2x^2 + 3x - 2)$ is multiplied out and simplified, find (i) the coefficient of x^3 (ii) the coefficient of x^2 (iii) the coefficient of x.

7. When the expression $(x-3)(x+2)(2x-3)$ is multiplied out and simplified, find (i) the coefficient of x^2 (ii) the coefficient of x.

8. The graph of $y = (x-2)(2x-3)(x+1)$ cuts the coordinate axes at which of the points below?

(2, 0)	(-2, 0)	(1.5, 0)	(-1.5, 0)	(3, 0)
(1, 0)	(-1, 0)	(0, 6)	(0, -6)	

9. The equation of the graph below could be

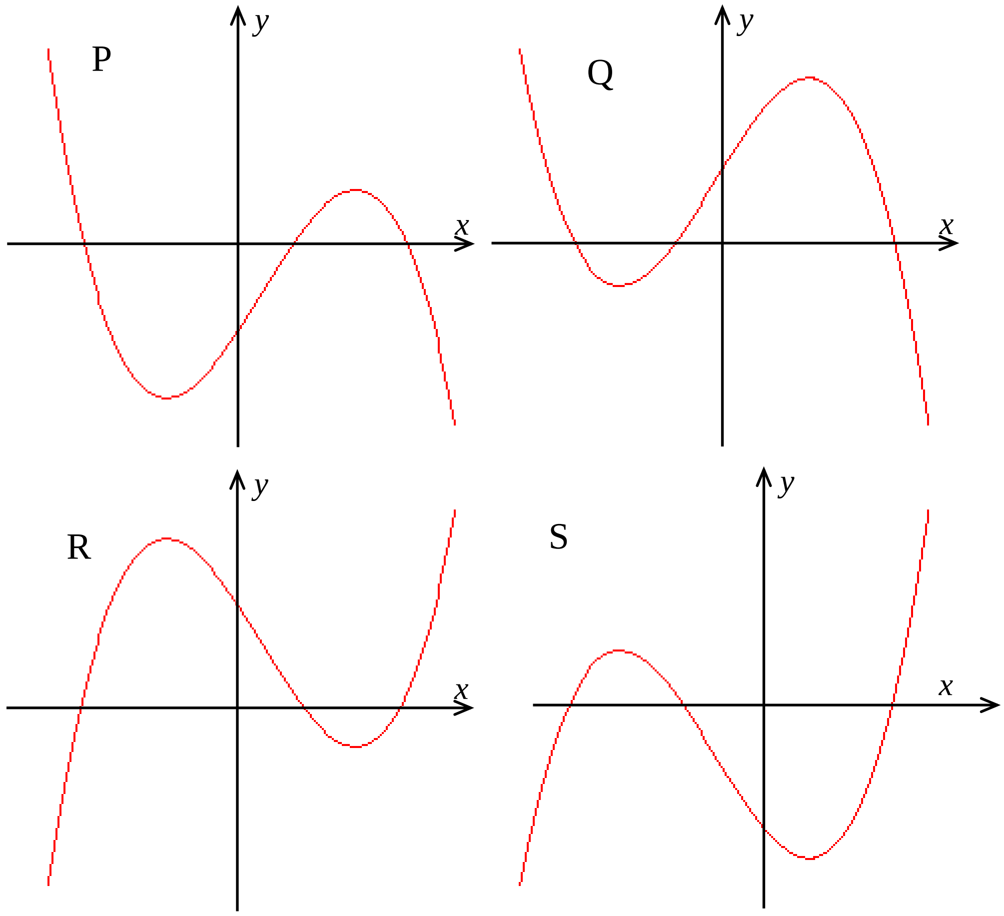


(a) $y = (x+1)(x-1)^2$ (b) $y = -2(x+1)(x-1)^2$

(c) $y = (x-1)(x+1)^2$ (d) $y = 2(x+1)(x-1)$

OCR AS Maths Polynomial 1 section test solutions

10. Which of the graphs below represents $y = (x - a)(x - b)(x + c)$, where a, b and c are all positive constants?



(a) P
(b) Q
(c) R
(d) S

Section B:

Quadratics – graphs and equations

1. Notes and examples on the following 8 pages of this document.

2. Helpful video tutorials for this topic:

Hegarty Maths Video Numbers Factorising Quadratics: 223-228; to solve: 230-233 Completing the Square: 235-237; to solve: 238-239 Quadratic Graphs: 252-256
Corbett Maths Video Numbers Factorising: 118, 119; to solve: 266 Completing the square: 10; to solve: 267a Quadratic Graphs: 264, 265, 267c
TL Maths https://sites.google.com/site/tlmaths314/home/gcse-to-a-level-maths-bridging-the-gap This link has all bridging the gap videos on – for this section: • Third factorising video

3. Pages 5 - 7 of the additional Algebra booklet:
   a. Exercises 1 and 2 provide opportunities to practise these skills.
   b. Exercise 3 is a challenge exercise.
   c. Answers for review are at the back of that booklet.

4. Additional sources of support:
   - CGP guide – "Head Start to A Level Maths"

5. Section test on page **24** of this booklet, after the notes and examples (worked solutions for review at the end of booklet).

OCR AS Mathematics Quadratic functions

Section 1: Quadratic graphs and equations

Notes and Examples

These notes contain subsections on

Factorising quadratic expressions

This should be revision of GCSE work. It is essential that you are confident in factorisation.

Example 1
Factorise the expressions
(i) x^2+4x+3
(ii) $x^2-4x-12$
(iii) $2x^2-7x+6$

Solution

(i) $x^2+4x+3=(x\,......)(x\,.....)$

Start with an x in each bracket

$x^2+4x+3=(x+1)(x+3)$

You need two numbers whose sum is 4 and whose product is 3. These are +1 and +3.

(ii) $x^2-4x-12=(x\,......)(x\,.....)$

Start with an x in each bracket

$x^2-4x-12=(x+6)(x+2)$

You need two numbers whose sum is –4 and whose product is –12. These are –6 and +2.

(iii) $2x^2-7x+6=(2x\,.....)(x\,.....)$

In this case you need to start with $2x$ in one bracket and x in the other.

$2x^2-7x+6=(2x-3)(x-2)$

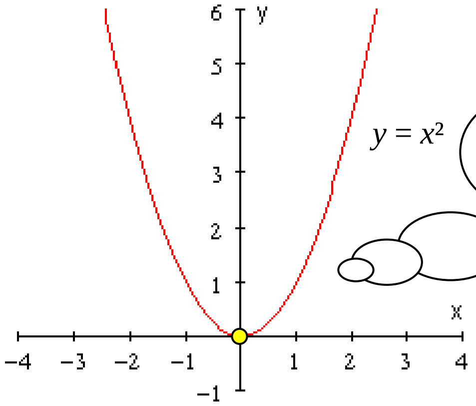
It is not so straightforward to find the two numbers in this case, because of the $2x$ in one bracket. The two numbers must have a product of +6, and as the coefficient of x is negative, they must both be negative. Try the different possibilities (–1 and –6, or –2 and –3, in either order), until you find the correct one.

Try the ***Quadratic factors walkthrough***.

Graphs of quadratic functions

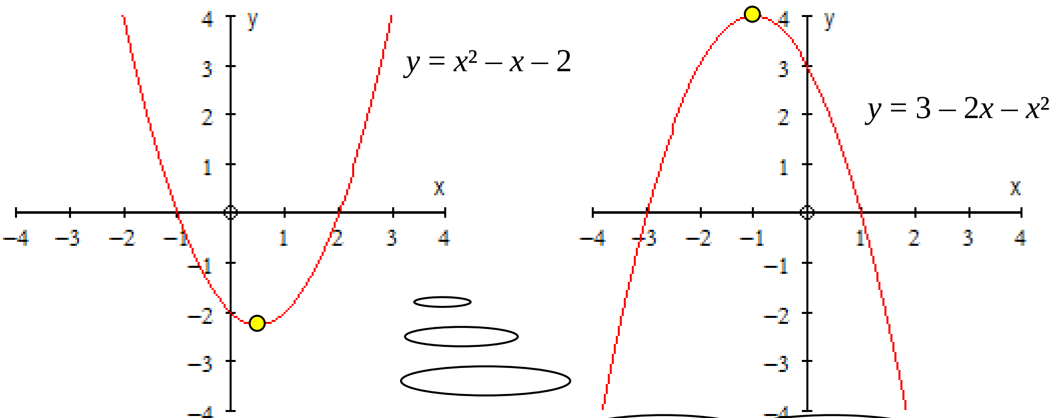
Factorising a quadratic expression gives you information about the graph of a quadratic function. Do not think of this work as just algebraic manipulation, think about it also in terms of the graph of the function. Linking algebra and graphs is a very important mathematical skill; the good news is that being able to consider problems both algebraically and graphically usually makes them easier! A graphical calculator or graphing software such as GeoGebra or Desmos will be very useful.

You may already be familiar with the graph of the simplest quadratic function, $y = x^2$.



The curve given by graphs of quadratics is called a ***parabola.*** Notice that all quadratic graphs have reflection symmetry. The mirror line is always a vertical line through the **turning point** or **vertex** of the curve (shown in yellow on these graphs).

All other quadratic graphs have basically the same shape, but they may be "stretched", "squashed", shifted or inverted.



Factorise the equations of these graphs. What is the relationship between the factorised form and the graph? Can you explain this?

Notice that the graphs of functions with a negative x^2 term are inverted (upside down).

Try the ***Quadratic graphs skill pack***.

Solving quadratic equations by factorisation

Solving quadratic equations is important not just from the algebraic point of view, but because it gives you information about the graph of a quadratic function. The roots of the equation $ax^2+bx+c=0$ tell you where the graph of the function $y=ax^2+bx+c$ crosses the x-axis, since these are the points where $y=0$.

Some quadratic equations can be solved by factorising.

Example 2

Solve these quadratic equations by factorising.

(i) $x^2+2x-8=0$ (ii) $2x^2+11x+12=0$

Solution

(i)

$$x^2+2x-8=0$$

$$(x+4)(x-2)=0$$

$$x+4=0 \text{ or } x-2=0$$

$$x=-4 \text{ or } x=2$$

For this expression to be zero, one or other of the factors must be zero.

(ii)

$$2x^2+11x+12=0$$

$$(2x+3)(x+4)=0$$

$$2x+3=0 \text{ or } x+4=0$$

$$x=-\tfrac{3}{2} \text{ or } x=-4$$

Try the ***Quadratic factors walkthrough*** and the ***Solving quadratics by factorisation skill pack***.

Quadratic equations in disguise

Some equations don't immediately look like quadratic equations, but they can be rewritten in quadratic form. Sometimes a substitution can be useful.

Example 3

Solve the equation $x^4-x^2-6=0$.

Solution

By putting $x^2 = y$ you can make this into a quadratic equation.

$y^2 - y^2 - 6 = 0$

$(y-3)(y+2) = 0$

$y = 3$ or $y = -2$

$x^2 = 3$ or $x^2 = -2$

$x = \pm\sqrt{3}$

Now change back to using x

Notice that $x^2 = -2$ has no real roots, so the only roots come from $x^2 = 3$

The turning point of a quadratic graph

A quadratic function is usually written in the form $y = ax^2 + bx + c$, where a, b and c are constants. However, writing quadratic functions in different forms can sometimes give you additional information about the function.

You have already seen that writing a quadratic function in factorised form gives you some useful information about the graph of the function. It tells you where the graph crosses the x-axis. This also applies to other polynomial functions.

However, sometimes you may not be interested in where the graph cuts the axes, but you may want to know the coordinates of the maximum or minimum point of the graph (often called the **vertex**). One way to do this is by using the *completed square form* for a quadratic function. This means the form $a(x-p)^2 + q$, so that you have a perfect square involving x, and a constant term.

To help you to visualise how the completed square form relates to the graph of a quadratic function, have a look at the ***Explore: Completing the square*** resource.

Example 4

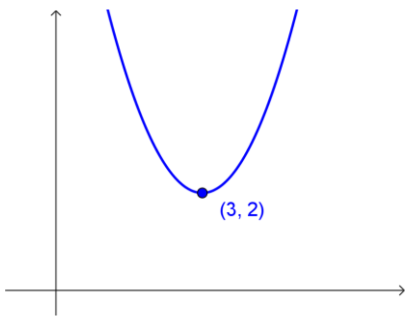
For each of the following quadratic graphs, write down the equation of the line of symmetry of the graph and the coordinates of the vertex (turning point), and hence sketch the graph.

(i) $y = (x-3)^2 + 2$ (ii) $y = (2x-1)^2 - 5$ (iii) $y = 1 - (x+2)^2$

Solution

(i) $y = (x-3)^2 + 2$

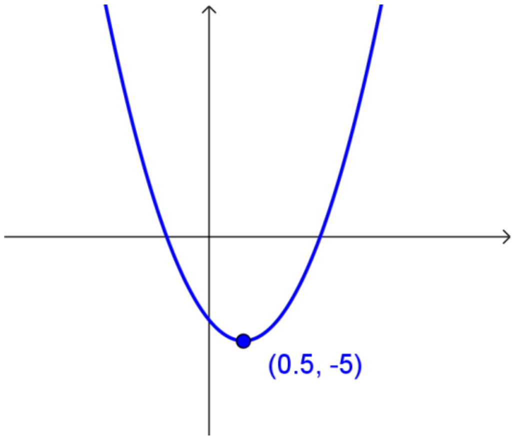
Line of symmetry is $x = 3$.
Minimum point is (3, 2).



(ii) $y = (2x-1)^2 - 5$

Line of symmetry is $x = \frac{1}{2}$.

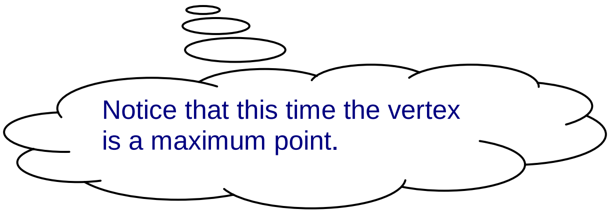
Minimum point is ($\frac{1}{2}$, -5).

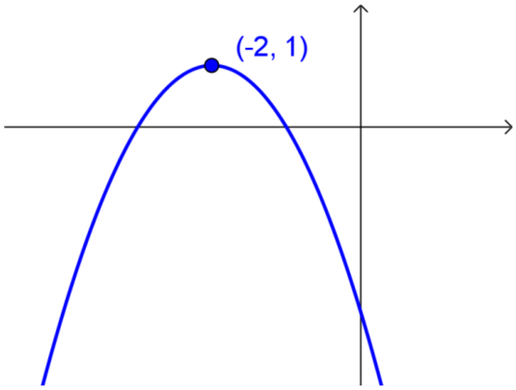


(iii) $y = 1-(x+2)^2$

Line of symmetry is x = -2.

Maximum point is (-2, 1).





Example 5

Find the equations of quadratic graphs with the given turning points. Give the equations in the form $y = ax^2 + bx + c$.

(i) Minimum point (1, -2)

(ii) Minimum point (-3, 1)

(iii) Maximum point (4, 3)

Solution

(i) The equation of the graph is
$$\begin{aligned} y &= (x-1)^2 - 2 \\ &= x^2 - 2x + 1 - 2 \\ &= x^2 - 2x - 1 \end{aligned}$$

(ii) The equation of the graph is
$$\begin{aligned} y &= (x+3)^2 + 1 \\ &= x^2 + 6x + 9 + 1 \\ &= x^2 + 6x + 10 \end{aligned}$$

(iii) The equation of the graph is
$$\begin{aligned} y &= 3-(x-4)^2 \\ &= 3-\left(x^2 - 8x + 16\right) \\ &= 3 - x^2 + 8x - 16 \\ &= -x^2 + 8x - 13 \end{aligned}$$

OCR AS Maths Quadratics 1 Notes and Examples

Completing the square

The examples below show how to write a quadratic function in the completed square form.

Example 6

Write the expression $x^2 + 4x + 7$ in the completed square form.

Solution

First you need to find a quadratic expression which is a perfect square and which begins with $x^2 + 4x$.
You do this by looking at the coefficient of x, in this case 4, and halving it. In this case you get 2.
This tells you that the perfect square you need is $(x + 2)^2$.

$$(x+2)^2 = x^2 + 4x + 4$$
$$x^2 + 4x + 7 = x^2 + 4x + 4 + 3$$
$$= (x+2)^2 + 3$$

$(x+2)^2$ is the 'square'

This is why the technique is called 'completing the square'.

There are several different approaches to writing out the working. They are all basically the same, so if you have learnt a different way which suits you, then stick to it.

The next example shows a situation where the coefficient of x^2 is not 1.

Example 7

Write the expression $2x^2 - 6x + 1$ in the completed square form.

Solution

Start by taking out the coefficient of x^2, in this case 2, as a factor.

$$2x^2 - 6x + 1 = 2[x^2 - 3x] + 1$$

Now look at the expression inside the square bracket. You need to find a quadratic expression which is a perfect square and starts with $x^2 - 3x$. Take the coefficient of x, which is –3, and halve it to get $-\frac{3}{2}$. The perfect square you need is therefore $\left(x - \frac{3}{2}\right)^2$.

$$\left(x-\tfrac{3}{2}\right)^2 = x^2 - 3x + \tfrac{9}{4} \text{ so } x^2 - 3x = \left(x-\tfrac{3}{2}\right)^2 - \tfrac{9}{4}$$

$$\begin{aligned} 2x^2 - 6x + 1 &= 2[(x-\tfrac{3}{2})^2 - \tfrac{9}{4}] + 1 \\ &= 2(x-\tfrac{3}{2})^2 - \tfrac{9}{2} + 1 \\ &= 2(x-\tfrac{3}{2})^2 - \tfrac{7}{2} \end{aligned}$$

In the next example, the coefficient of x^2 is negative. This can be dealt with by taking out a factor -1.

Example 8

Write the expression $5 + x - x^2$ in the completed square form.

Hence sketch the graph of $y = 5 + x - x^2$, showing the coordinates of its turning point.

Solution

Start by taking out –1 as a factor from the first two terms.

$$5 + x - x^2 = -[x^2 - x] + 5$$

Now you need a quadratic expression which is a perfect square and starts with $x^2 - x$. Half the coefficient of x is $-\tfrac{1}{2}$, so the perfect square you need is $(x-\tfrac{1}{2})^2$.

$$(x-\tfrac{1}{2})^2 = x^2 - x + \tfrac{1}{4} \text{ so } x^2 - x = (x-\tfrac{1}{2})^2 - \tfrac{1}{4}$$

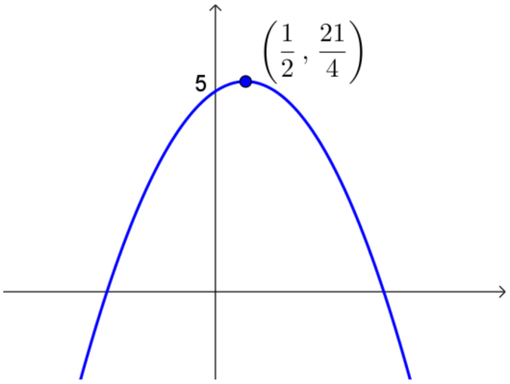
$$\begin{aligned} 5 + x - x^2 &= -[(x-\tfrac{1}{2})^2 - \tfrac{1}{4}] + 5 \\ &= -(x-\tfrac{1}{2})^2 + \tfrac{1}{4} + 5 \\ &= -(x-\tfrac{1}{2})^2 + \tfrac{21}{4} \end{aligned}$$

$y = 5 + x - x^2$ can therefore be written as $y = -(x-\tfrac{1}{2})^2 + \tfrac{21}{4}$.

Since the coefficient of x^2 is negative, the graph has a maximum point rather than a minimum point.

From the completed square form, the graph has maximum point $\left(\tfrac{1}{2}, \tfrac{21}{4}\right)$.

Also note that it passes through the point (0, 5).



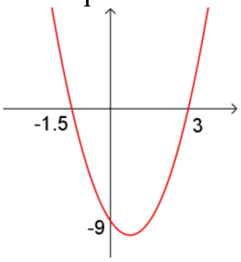
Try the ***Completing the square walkthrough*** and the ***Completing the square skill pack***.

Section 1: Quadratic graphs and equations

Section test

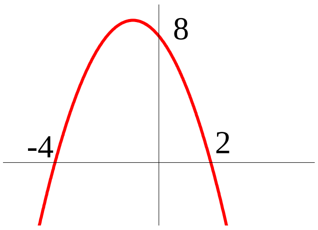
1. Factorise the expression $4y^2 + 5y - 6$.

2. The equation of the graph below is given by



(a) $y = 2x^2 + 3x - 9$ (b) $y = 2x^2 - 3x - 9$
(c) $y = x^2 - 1.5x - 4.5$ (d) $y = x^2 + 1.5x - 4.5$

3. The equation of the graph below is given by



(a) $y = x^2 - 2x - 8$ (b) $y = -x^2 - 2x + 8$
(c) $y = -x^2 + 2x - 8$ (d) $y = x^2 + 2x - 8$

4. The quadratic expression $x^2 - 2x - 3$ can be written in the form $(x+a)^2 + b$. Find the values of *a* and *b*.

5. The quadratic expression $3 + x - x^2$ can be written in the form $b - (x+a)^2$. Find the values of *a* and *b*.

6. The quadratic expression $2x^2 + 6x + 1$ can be written in the form $a(x+b)^2 + c$. Find the values of *a*, *b* and *c*.

7. Find the equation of a quadratic graph with minimum point (1, -4).

8. Find the equation of a quadratic graph with maximum point (-2, 5).

9. Find the coordinates of the vertex of the graph of $y = x^2 - 2x - 1$. State whether the vertex is a maximum or a minimum point.

10. Find the coordinates of the vertex of the graph of $y = -x^2 + 5x + 2$. State whether the vertex is a maximum or a minimum point.

Section C:

Quadratics – formula

1. Notes and examples on the following 5 pages of this document.

2. Helpful video tutorials for this topic:

Hegarty Maths Video Numbers Solving with the formula: 241-242 The discriminant: 243
Corbett Maths Video Numbers Solving with the formula: 267 Quadratic Formula Proof: 267b

3. Pages 11 - 13 of the additional Algebra booklet:
    a. Exercises 1 and 2 provide opportunities to practise these skills.
    b. Exercise 3 is a challenge exercise.
    c. Answers for review are at the back of that booklet.

4. Additional sources of support:
    - CGP guide – “Head Start to A Level Maths”

5. Section test on pages **30-31** of this booklet, after the notes and examples (worked solutions for review at the end of booklet).

Section 2: The quadratic formula

Notes and Examples

These notes contain subsections on
- Solving quadratic equations using the formula
- Problem solving

Solving quadratic equations using the formula

If a quadratic equation is written in the completed square form, it is easy to solve.

Example 1
(i) Write $x^2 - 4x - 5$ in the completed square form.
(ii) Hence solve the equation $x^2 - 4x - 5 = 0$

Solution
(i) $x^2 - 4x - 5 = (x-2)^2 - 4 - 5$
$\qquad\qquad\qquad = (x-2)^2 - 9$

(ii) $(x-2)^2 = 9$
$x - 2 = \pm 3$
$x = 2 \pm 3$
$x = 5$ or -1

However, unless you already have the equation in the completed square form, as in the example above, it is easier to use the quadratic formula, which is just a generalisation of the completing the square method.

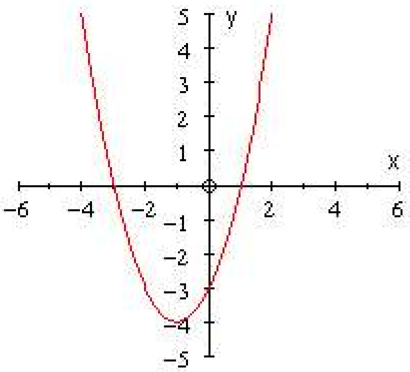
The quadratic formula for the solutions of the equation $ax^2 + bx + c = 0$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

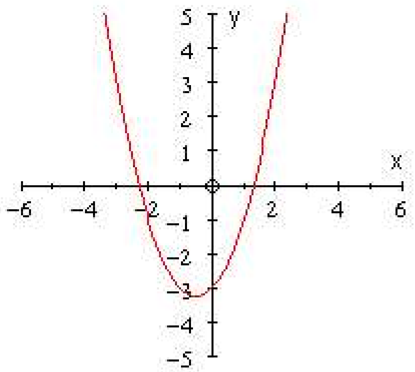
The expression $b^2 - 4ac$ is called the **discriminant.** This is very important as it tells you something about the nature of the solutions.
In each case the solution(s) correspond to the points where the graph meets the x-axis.

- If the discriminant is positive, then there are two real solutions. (If the discriminant is a positive square number, then the two real solutions are rational and it is possible to solve the equation by factorisation; otherwise the solutions are irrational and you must use the quadratic formula.)

OCR AS Maths Quadratics 2 Notes and Examples

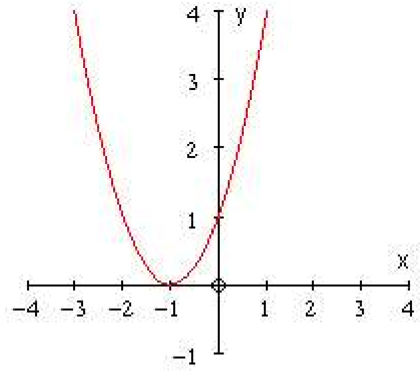


$y = x^2 + 2x - 3$
Discriminant = 16
Two rational solutions



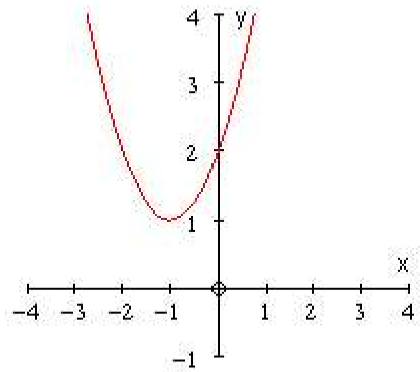
$y = x^2 + x - 3$
Discriminant = 13
Two real, irrational solutions

- If the discriminant is zero, then the quadratic is a perfect square and there is one real solution, which can be found by factorisation.



$y = x^2 + 2x + 1$
Discriminant = 0
One real solution

- If the discriminant is negative, then there are no real solutions.



$y = x^2 + 2x + 2$
Discriminant = -4
No real solutions

As the graph does not meet the x-axis, there cannot be any real solutions.

When you need to solve a quadratic equation, it is useful to quickly work out the discriminant before you start, so that you know whether there are real solutions, and whether the equation can be solved by factorisation.

Your calculator may be able to solve quadratic equations, and some calculators will give the answers in exact form (using surds). However, you may be required to show working in some questions, so you must know the quadratic formula and be confident in using it.

Example 2
For each of the following quadratic equations, find the discriminant and solve the equation, where possible, by a suitable method

(i) $2x^2-4x+1=0$ (ii) $6x^2+11x-10=0$
(iii) $3x^2-2x+4=0$ (iv) $4x^2+12x+9=0$

Solution

(i) $a = 2$, $b = -4$, $c = 1$

Discriminant $= (-4)^2-4\times2\times1=16-8=8$

Since the discriminant is positive, there are two real solutions. As it is not a square number, the equation must be solved using the quadratic formula.

$$\begin{aligned} x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ &= \frac{4\pm\sqrt{8}}{2\times2} \\ &= \frac{4\pm2\sqrt{2}}{4} \\ &= \frac{2\pm\sqrt{2}}{2} \end{aligned}$$

(ii) $a = 6$, $b = 11$, $c = -10$

Discriminant $= 11^2-4\times6\times-10=121+240=361$

Since the discriminant is positive, there are two real solutions. As it is a square number (19^2), the equation can be solved by factorisation.

$$6x^2+11x-10=0$$
$$(3x-2)(2x+5)=0$$
$$x=\tfrac{2}{3} \text{ or } x=-\tfrac{5}{2}$$

(iii) $a = 3$, $b = -2$, $c = 4$

Discriminant $= (-2)^2-4\times3\times4=4-48=-44$

Since the discriminant is negative, there are no real solutions.

(iv) $a = 4$, $b = 12$, $c = 9$

Discriminant $= 12^2-4\times4\times9=144-144=0$

Since the discriminant is zero, there is one solution and the equation can be solved by factorisation into a perfect square.

$$4x^2+12x+9=0$$
$$(2x+3)^2=0$$
$$x=-\tfrac{3}{2}$$

Try the ***Quadratic formula walkthrough***, and the ***Quadratic formula skill pack***.

Problem solving

Some problems, when translated into algebra, involve quadratic equations.

Example 3
A rectangular box has width 2 cm greater than its length, and height 3 cm less than its length. The total surface area of the box is 548 cm².
What are the dimensions of the box?

Solution
Let the length of the box be x cm.
The width of the box is $x + 2$ cm, and the height is $x - 3$ cm.

The surface area of the box is given by $2x(x+2)+2x(x-3)+2(x+2)(x-3)$.

$$2x(x+2)+2x(x-3)+2(x+2)(x-3)=548$$
$$x(x+2)+x(x-3)+(x+2)(x-3)=274$$

Divide through by 2

$$x^2+2x+x^2-3x+x^2-x-6=274$$
$$3x^2-2x-280=0$$

The discriminant is 3364, which is 58², so this must factorise

$$(3x+28)(x-10)=0$$
$$x=10$$

$3x + 28 = 0$ gives a negative value of x, which does not make sense in this context. So the solution must be $x - 10 = 0$.

The length of the box is 10 cm, the width is 12 cm and the height is 7 cm.

Notice that in Example 3, you could discard one of the possible solutions as a negative solution did not make sense in the context. This is not always the case. In some situations, a negative solution can have a practical meaning. For example if the height of a stone thrown from the edge of a cliff is negative, this simply means that the stone is below the level of the cliff at that point. However, if the stone was thrown from level ground, then a negative height does not make sense.

Some problems leading to quadratic equations do have two possible solutions. Always consider whether your solution(s) make sense in the context.

Section 2: The quadratic formula

Section test

1. Find the discriminant of the quadratic equation $2x^2+5x-1=0$.

2. The quadratic equation in Question 1 has

(a) one real root
(b) two rational roots
(c) two real irrational roots
(d) no real roots

3. Which of the quadratic equations below do not have real roots? Choose as many as apply.

(i) $x^2+3x+1=0$
(ii) $2x^2-3x+4=0$
(iii) $3x^2+x-2=0$

4. The roots of the equation $x^2+2x-5=0$ are

(a) $-1\pm\sqrt{24}$
(b) $-1\pm\sqrt{6}$
(c) $-1\pm\sqrt{12}$
(d) There are no real roots

5. The roots of the equation $2x^2-11x+15=0$ are

(a) 2.5 and 3
(b) 1.5 and 5
(c) $\dfrac{11\pm\sqrt{241}}{4}$
(d) There are no real roots

6. The roots of the equation $3x^2-2x+4=0$ are

(a) 2 and $\frac{2}{3}$
(b) $\dfrac{1\pm\sqrt{11}}{3}$
(c) $\dfrac{1\pm\sqrt{13}}{3}$
(d) There are no real roots

7. The roots of the equation $2x^2-5x-4=0$ are

(a) $\dfrac{5\pm\sqrt{57}}{4}$
(b) $\dfrac{-5\pm\sqrt{57}}{4}$
(c) $\dfrac{5\pm\sqrt{7}}{4}$
(d) $\dfrac{-5\pm\sqrt{7}}{4}$

8. The quadratic equation $x^2+kx+2k-3=0$ has equal roots.
The possible value(s) of k are

(a) 2 or 6
(b) 2 only
(c) 3 or 4
(d) 0 only

OCR AS Maths Quadratics 2 section test solutions

9. How many real roots does the equation $x^4 + 6x^2 + 4 = 0$ have?

10. An object is thrown vertically upwards so that its height h metres above the ground at time t seconds is given by $h = 20t - 5t^2 + 1$. After how many seconds does it hit the ground? Give your answer correct to 2 d.p.

Section D:

<u>Simultaneous Equations</u>

1. Notes and examples on the following 4 pages of this document.

2. Helpful video tutorials for this topic:

<u>Hegarty Maths Video Numbers</u> Simultaneous Equations: 190-195 Quadratic Simultaneous Equations: 246 Using Graphs: 259
<u>Corbett Maths Video Numbers</u> Simultaneous Equations: 295, 296 Quadratic Simultaneous Equations: 298

3. Pages 11 - 13 of the additional Algebra booklet:
   a. Exercises 1 and 2 provide opportunities to practise these skills.
   b. Exercise 3 is a challenge exercise.
   c. Answers for review are at the back of that booklet.

4. Additional sources of support:
   - CGP guide – "Head Start to A Level Maths"

5. Section test on pages **<u>37-38</u>** of this booklet, after the notes and examples (worked solutions for review at the end of booklet).

Section 1: Simultaneous equations

Notes and Examples

These notes contain subsections on

- Linear simultaneous equations using elimination
- Linear simultaneous equations using substitution
- One linear and one quadratic equation
- The intersection of a line and a curve

Linear simultaneous equations using elimination

This work is revision of GCSE. You need to make sure that you can solve linear simultaneous equations confidently before you move on to the work on one linear and one quadratic equation, which will probably be new to you.

Simultaneous equations involve more than one equation and more than one unknown. To solve them you need the same number of equations as there are unknowns.

One method of solving simultaneous equations involves adding or subtracting multiples of the two equations so that one unknown disappears. This method is called *elimination*, and is shown in the next example.

Example 1

Solve the simultaneous equations

$$3p+q=5$$
$$p-2q=4$$

Solution

① $\quad 3p+q=5$

② $\quad p-2q=4$

> Adding or subtracting these equations will not eliminate either p or q. However, you can multiply the first equation by 2, and then add. This will eliminate p.

① × 2 $\quad 6p+2q=10$

② $\quad p-2q=4$

Adding: $\quad 7p=14$

$$p=2$$

$$3\times 2+q=5$$
$$6+q=5$$
$$q=-1$$

> Now substitute this value for p into one of the original equations – you can use either, but in this example, ① is used.

The solution is $p = 2$, $q = -1$.

Notice that, in Example 1, you could have multiplied equation ② by 3 and then subtracted. This would give the same answer.

Sometimes you need to multiply each equation by a different number before you can add or subtract. This is the case in the next example.

Example 2
5 pencils and 2 rubbers cost £1.50
8 pencils and 3 rubbers cost £2.35
Find the cost of a pencil and the cost of a rubber.

Solution

① $5p + 2r = 150$

② $8p + 3r = 235$

Let p represent the cost of a pencil and r represent the cost of a rubber. It is easier to work in pence.

① × 3 $15p + 6r = 450$

② × 2 $16p + 6r = 470$

The easiest method is to multiply equation ① by 3 and equation ② by 2. (You could of course multiply ① by 8 and ② by 5).

Subtracting: $-p = -20$

$p = 20$

$$5 \times 20 + 2r = 150$$

Substitute this value of p into equation ①

$$100 + 2r = 150$$
$$2r = 50$$
$$r = 25$$

A pencil costs 20p and a rubber costs 25p.

Linear simultaneous equations using substitution

An alternative method of solving simultaneous equations is called substitution. This can be the easier method to use in cases where one equation gives one of the variables in terms of the other. This is shown in the next example.

Example 3
Solve the simultaneous equations

$$3x - 2y = 11$$
$$y = 5 - 2x$$

Solution

Substitute the expression for y given in the second equation, into the first equation.

$$\begin{aligned} 3x - 2(5 - 2x) &= 11 \\ 3x - 10 + 4x &= 11 \\ 7x &= 21 \\ x &= 3 \end{aligned}$$

Multiply out the brackets

$$\begin{aligned} y &= 5 - 2 \times 3 \\ &= 5 - 6 \\ &= -1 \end{aligned}$$

Substitute the value for x into the original second equation

The solution is $x = 3$, $y = -1$

One linear and one quadratic equation

When you need to solve a pair of simultaneous equations, one of which is linear and one of which is quadratic, you need to substitute the linear equation into the quadratic equation.

Example 4

Solve the simultaneous equations

$$x^2 + 2y^2 = 6$$
$$x - y = 1$$

Solution

Start by using the linear equation to write one variable in terms of the other.

$$x = y + 1$$

Now substitute this expression for y into the first equation

$$\begin{aligned} (y+1)^2 + 2y^2 &= 6 \\ y^2 + 2y + 1 + 2y^2 &= 6 \\ 3y^2 + 2y - 5 &= 0 \\ (3y+5)(y-1) &= 0 \\ y = -\tfrac{5}{3} \text{ or } y &= 1 \end{aligned}$$

Multiply out, simplify and factorise

Sometimes you will need to use the quadratic formula to solve the resulting quadratic equation.

$y = -\frac{5}{3}$ $\quad$ $x = y + 1 = -\frac{5}{3} + 1 = -\frac{2}{3}$

$y = 1$ $\quad$ $x = y + 1 = 1 + 1 = 2$

Now substitute each value for y into the linear equation to find the corresponding values of x

The solutions are $x = -\frac{2}{3}, y = -\frac{5}{3}$ and $x = 2, y = 1$

The intersection of a line and a curve

Just as the point of intersection of two straight lines can be found by solving the equations of the two lines simultaneously, the point(s) of intersection of a line and a curve can be found by solving their equations simultaneously.

In many cases, the equations of both the line and the curve are given as an expression for y in terms of x. When this is the case, a sensible first step is to equate the expressions for y, as this leads to an equation in x only.

Example 5
Find the coordinates of the points where the line $y = x + 2$ meets the curve $y = x^2$ $3x + 5$.

Solution

$$x^2 - 3x + 5 = x + 2$$
$$x^2 - 4x + 3 = 0$$
$$(x - 3)(x - 1) = 0$$
$$x = 3 \text{ or } x = 1$$

Equate the expressions for y to give an equation in x only

When $x = 3$ then $y = 3 + 2 = 5$
When $x = 1$ then $y = 1 + 2 = 3$.

Substitute the x values into the equation of the line

The points where the line meets the curve are (3, 5) and (1, 3).

You should check that each of these points satisfies the equation of the curve. (You have already used the equation of the line to find the y-values).

Notice that this problem involved solving a quadratic equation, which in this case had two solutions, showing that the line crossed the curve twice. However, the quadratic equation could have had no solutions, which would indicate that the line did not meet the curve at all, or one repeated solution, which would indicate that the line touches the curve.

Section 1: Simultaneous equations

Section test

1. Solve the simultaneous equations
$$x + 3y = 5$$
$$3x - y = 5$$

2. Solve the simultaneous equations
$$7a - 3b = 6$$
$$3a - 2b = 5$$

3. For the simultaneous equations
$$5a + 7b = 17$$
$$a = 1 - 3b$$
what is the correct value of a for the solution?

4. For the simultaneous equations
$$2x = 5y - 2$$
$$6y = 1 + 4x$$
what is the correct value of x for the solution?

5. For the simultaneous equations
$$s^2 + 2t^2 = 6$$
$$3s - t = 5$$
find the values of t for the solutions.

6. Find the x-coordinates of the points of intersection between the line $y = 2x + 3$ and the curve $y = x^2 - x - 1$.

7. For the simultaneous equations
$$x^2 + 2y = 5$$
$$2x - 3y = 12$$
find the values of y for the solutions.

8. The solutions of the simultaneous equations
$$x^2 + 2xy + 1 = 0$$
$$y + x = 1$$
are

(a) $x = 1$, $y = 0$ and $x = -1$, $y = 2$

(b) $x = 1 + \sqrt{2}, y = \sqrt{2}$ and $x = 1 - \sqrt{2}, y = -\sqrt{2}$

(c) $x = 1 + \sqrt{2}, y = -\sqrt{2}$ and $x = 1 - \sqrt{2}, y = \sqrt{2}$

(d) $x = 1 + \sqrt{8}, y = -\sqrt{8}$ and $x = 1 - \sqrt{8}, y = \sqrt{8}$

9. The solutions of the simultaneous equations

$$2x^2 + y^2 = 21$$

$$y = 2x - 3$$

are

(a) $x = 1+\sqrt{3}, y = -1+2\sqrt{3}$ and $x = 1-\sqrt{3}, y = -1-2\sqrt{3}$

(b) $x = 1+\sqrt{12}, y = -1+2\sqrt{12}$ and $x = 1-\sqrt{12}, y = -1-2\sqrt{12}$

(c) $x = 1+\sqrt{3}, y = 2\sqrt{3}-3$ and $x = 1-\sqrt{3}, y = 2\sqrt{3}-3$

(d) $x = 1+\sqrt{12}, y = 2\sqrt{12}-3$ and $x = 1-\sqrt{12}, y = 2\sqrt{12}-3$

10. The line $y = 2x + k$ and the curve $y = x^2 - 4x + 2$ have just one point of contact. Find the value of k.

Section E:

Inequalities

1. Notes and examples on the following 5 pages of this document.

2. Helpful video tutorials for this topic:

Hegarty Maths Video Numbers Linear Inequalities: 267-271 Quadratic Inequalities: 277
Corbett Maths Video Numbers Linear Inequalities: 176-179 Quadratic Inequalities: 378

3. Pages 14 - 16 of the additional Algebra booklet:
    a. Exercises 1 and 2 provide opportunities to practise these skills.
    b. Exercise 3 is a challenge exercise.
    c. Answers for review are at the back of that booklet.

4. Additional sources of support:
    - CGP guide – “Head Start to A Level Maths”

5. Section test on page **45** of this booklet, after the notes and examples (worked solutions for review at the end of booklet).

Section 2: Inequalities

Notes and Examples

These notes contain subsections on

Inequalities

Inequalities are similar to equations, but instead of an equals sign, =, they involve one of these signs:

$<$	less than
$>$	greater than
$\leq$	less than or equal to
$\geq$	greater than or equal to

This means that whereas the solution of an equation is a specific value, or two or more specific values, the solution of an inequality is a range of values.

Inequalities can be solved in a similar way to equations, but you do have to be very careful, as in some situations you need to reverse the inequality. This is shown in these examples.

Linear inequalities

A linear inequality involves only terms in x and constant terms.

Example 1

Solve the inequality $3x+1>x-5$

You can treat this just like a linear equation.

Solution

$3x+1>x-5$

Subtract x from each side

$2x+1>-5$

Subtract 1 from each side

$2x>-6$

Divide both sides by 2

$x>-3$

The next example involves a situation where you have to divide by a negative number. When you are solving an equation, multiplying or dividing by a negative number is not a problem. However, things are different with inequalities.

OCR AS Maths Equations 2 Notes and Examples

The statement $-3 < 2$ is clearly true.

If you add something to each side, it is still true

$-3 + 1 < 2 + 1$
$-2 < 3$ ✓

If you subtract something from each side, it is still true

$-3 - 4 < 2 - 4$
$-7 < -2$ ✓

If you multiply or divide each side by a positive number, it is still true

$-3 \times 2 < 2 \times 2$
$-6 < 4$ ✓

However, if you multiply each side by a negative number then things go wrong!

$-3 \times -2 < 2 \times -2$
$6 < -4$ ✗

When you multiply or divide each side by a negative number, you must reverse the inequality.

The following example demonstrates this. Two solutions are given: in the first the inequality is reversed when dividing by a negative number, in the second this situation is avoided by a different approach.

Example 2
Solve the inequality $1 - x \geq 2x - 5$

Solution (1)

$$1 - x \geq 2x - 5$$

Subtract $2x$ from each side

$$1 - 3x \geq -5$$

Subtract 1 from each side

$$-3x \geq -6$$

Divide both sides by -3, reversing the inequality.

$$x \leq 2$$

Solution (2)

$$1 - x \geq 2x - 5$$

Add x to each side

$$1 \geq 3x - 5$$

Add 5 to each side

$$6 \geq 3x$$

Divide both sides by 3

$$2 \geq x$$

Finish by writing the inequality the other way round.

$$x \leq 2$$

You can check that you have the sign the right way round by picking a number within the range of the solution, and checking that it satisfies the original inequality. In the above example, you could try $x = 1$. In the original inequality you get $0 \geq -3$, which is correct.

Quadratic inequalities

You can solve a quadratic inequality by factorising the quadratic expression, just as you do to solve a quadratic equation. This tells you the boundaries of the solutions. The easiest way to find the solution is then to sketch a graph.

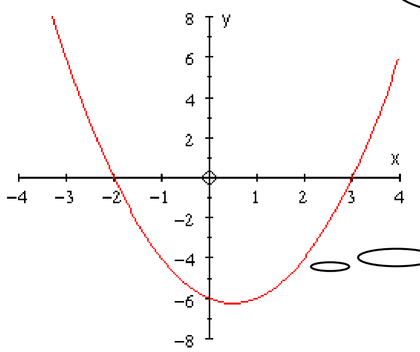
Example 3

Solve the inequality $x^2 - x - 6 < 0$

Solution

$$x^2 - x - 6 < 0$$
$$(x+2)(x-3) < 0$$

This shows that the graph of $y = x^2 - x - 6$ cuts the x-axis at $x = -2$ and $x = 3$. Use this information to sketch the graph.



The solution to the inequality is the negative part of the graph. This is the part between –2 and 3.

The solution is $\quad -2 < x < 3$

Notice that in Example 3, the solution is the set of values of x for which **both** $x > -2$ **and** $x < 3$. You can write this in set notation as

$$\{x : x > -2) \cap \{x : x < 3\}$$

The symbol $\cap$ is the 'intersection symbol' – so this means all the values of x that are in both sets

However, writing it in the form $-2 < x < 3$ is usually the clearest way to express the solution, as it shows that x lies between these two values.

Example 4

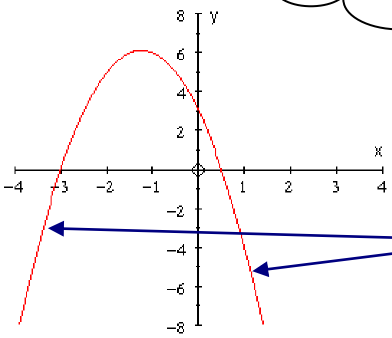
Solve the inequality $3-5x-2x^2 \le 0$

Solution

$$3-5x-2x^2 \le 0$$

$$(3+x)(1-2x) \le 0$$

This shows that the graph of $y=3-5x-2x^2$ cuts the x-axis at x = -3 and x = ½. You can now sketch the graph – note that as the term in x^2 is negative, the graph is inverted.



The solution to the inequality is the negative part of the graph. This is in fact two separate parts.

The solution is $x \le -3$ or $x \ge \frac{1}{2}$.

Notice the use of '**or**' in the solution. The value of x must be either less than or equal to 3, or greater than or equal to $\frac{1}{2}$ - it **cannot be both**, so the word 'and' must not be used. You cannot write this as a single inequality.

You can write the solution in set notation like this:

$$\{x : x \le -3\} \cup \{x : x \ge \tfrac{1}{2}\}$$

The symbol $\cup$ is the 'union' symbol and it means the combination of two sets, so it is all the values which are in either one set or the other.

Note: Example 4 involves a negative term in x^2. if you prefer to work with a positive x^2 term, you can change all the signs in the original inequality and reverse the inequality, giving $2x^2+5x-3 \ge 0$. The graph will then be the other way up, and you will take the positive part of the graph, so the solution will be the same.

For more practice, try the ***Inequalities skill pack***.

Section 2: Inequalities

Section test

Solve the inequality in each of questions 1 – 8.

1. $5x-2<3x+8$

2. $\dfrac{x+1}{2}\leq\dfrac{2-x}{3}$

3. $2(1-2x)-x>3(x+1)+7$

4. $x^2-5x+4>0$

5. $2x^2-x-3\leq 0$

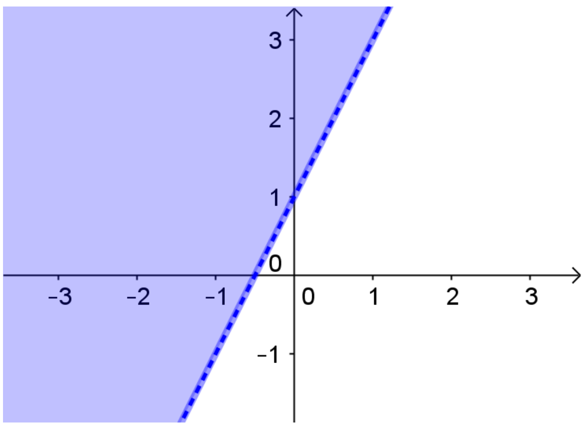
6. $8+2x-x^2<0$

7. $6x^2+x-2\geq 0$

8. $2x^2-x+1<x^2-4x-1$

9. The quadratic equation $x^2+(k-1)x+4=0$ has at least one real root.
   Find the range of possible values for *k*.

10. Which of the inequalities below describes the shaded region in the diagram?



(a) $y<2x+1$ (b) $y>2x+1$

(c) $y\leq 2x+1$ (d) $y\geq 2x+1$

SOLUTIONS TO SECTION REVIEW TESTS

OCR AS Maths Polynomial 1 section test solutions

Solutions to section test

1. $$\begin{aligned} f(x)+g(x) &= \left(x^4-2x^3+2x-1\right)+\left(3x^3+4x^2-2x+5\right) \\ &= x^4+\left(-2x^3+3x^3\right)+4x^2+(2x-2x)-1+5 \\ &= x^4+x^3+4x^2+4 \end{aligned}$$

2. $$\begin{aligned} f(x)-g(x) &= \left(x^4-2x^3+2x-1\right)-\left(3x^3+4x^2-2x+5\right) \\ &= x^4-2x^3+2x-1-3x^3-4x^2+2x-5 \\ &= x^4+\left(-2x^3-3x^3\right)-4x^2+(2x+2x)-1-5 \\ &= x^4-5x^3-4x^2+4x-6 \end{aligned}$$

3. $$\begin{aligned} (x-2)f(x) &= (x-2)\left(x^4-2x^3+2x-1\right) \\ &= x\left(x^4-2x^3+2x-1\right)-2\left(x^4-2x^3+2x-1\right) \\ &= x^5-2x^4+2x^2-x-2x^4+4x^3-4x+2 \\ &= x^5+\left(-2x^4-2x^4\right)+4x^3+2x^2+(-x-4x)+2 \\ &= x^5-4x^4+4x^3+2x^2-5x+2 \end{aligned}$$

4. $$\begin{aligned} (2x+1)g(x) &= (2x+1)\left(3x^3+4x^2-2x+5\right) \\ &= 2x\left(3x^3+4x^2-2x+5\right)+\left(3x^3+4x^2-2x+5\right) \\ &= 6x^4+8x^3-4x^2+10x+3x^3+4x^2-2x+5 \\ &= 6x^4+\left(8x^3+3x^3\right)+\left(-4x^2+4x^2\right)+(10x-2x)+5 \\ &= 6x^4+11x^3+8x+5 \end{aligned}$$

5. $f(x)$ has degree 4, and $g(x)$ has degree 3, so the highest power of the product of $f(x)$ and $g(x)$ will be 7. So the degree is 7.

6. $$\begin{aligned} &(x^2-x+1)(2x^2+3x-2) \\ &= x^2(2x^2+3x-2)-x(2x^2+3x-2)+1(2x^2+3x-2) \\ &= 2x^4+3x^3-2x^2-2x^3-3x^2+2x+2x^2+3x-2 \\ &= 2x^4+x^3-3x^2+5x-2 \end{aligned}$$

OCR AS Maths Polynomial 1 section test solutions

7. $(x-3)(x+2)(2x-3) = (x^2 - x - 6)(2x-3)$

$$= x^2(2x-3) - x(2x-3) - 6(2x-3)$$
$$= 2x^3 - 3x^2 - 2x^2 + 3x - 12x + 18$$
$$= 2x^3 - 5x^2 - 9x + 18$$

8. The graph cuts the x-axis when $(x-2)(2x-3)(x+1) = 0$

$x - 2 = 0 \quad \Rightarrow x = 2$

$2x - 3 = 0 \quad \Rightarrow x = \frac{3}{2}$

$x + 1 = 0 \quad \Rightarrow x = -1$

so the graph cuts the x-axis at (2, 0), (1.5, 0) and (-1, 0).
The graph cuts the y-axis where x = 0, so $y = -2 \times -3 \times 1 = 6$
so the graph cuts the y-axis at (0, 6).

9. The graph touches the x-axis at $x = 1$ so there is a repeated factor $(x-1)^2$. There is also a factor $(x+1)$, so the equation of the graph is of the form $y = a(x+1)(x-1)^2$.
When $x = 0$, $y = -2$, so $-2 = a(1)(-1)^2$

$$a = -2$$

The graph is therefore $y = -2(x+1)(x-1)^2$

10. The graph of $y = (x-a)(x-b)(x+c)$ crosses the x-axis at (a, 0), (b, 0) and (-c, 0).
Since two of the intersections are with the positive x-axis and one with the negative x-axis, the graph must be either P or R.
Since y is positive for large positive values of x, the correct graph is R.

OCR AS Maths Quadratics 1 section test solutions

Solutions to section test

1. $4 \times -6 = -24$, so need two numbers which multiply together to make -24, and add together to make 5. These are 8 and -3.

$$\begin{aligned} 4y^2 + 5y - 6 &= 4y^2 + 8y - 3y - 6 \\ &= 4y(y+2) - 3(y+2) \\ &= (4y-3)(y+2) \end{aligned}$$

2. The graph cuts the x-axis at -1.5 and 3, so the equation is of the form

$$\begin{aligned} y &= a(x+1.5)(x-3) \\ &= a(x^2 - 3x + 1.5x - 4.5) \\ &= a(x^2 - 1.5x - 4.5) \end{aligned}$$

When $x = 0$, $y = -9$, so $-9 = -4.5a$

$$a = 2$$

The equation of the graph is $y = 2(x^2 - 1.5x - 4.5)$

$$y = 2x^2 - 3x - 9$$

3. The graph cuts the x-axis at -4 and 2, so the equation is of the form

$$\begin{aligned} y &= a(x+4)(x-2) \\ &= a(x^2 - 2x + 4x - 8) \\ &= a(x^2 + 2x - 8) \end{aligned}$$

When $x = 0$, $y = 8$, so $8 = -8a$

$$a = -1$$

The equation of the graph is $y = -(x^2 + 2x - 8)$

$$y = -x^2 - 2x + 8$$

4. $$\begin{aligned} x^2 - 2x - 3 &= (x-1)^2 - 1 - 3 \\ &= (x-1)^2 - 4 \end{aligned}$$

5. $$\begin{aligned} 3 + x - x^2 &= 3 - (x^2 - x) \\ &= 3 - \left((x - \tfrac{1}{2})^2 - \tfrac{1}{4}\right) \\ &= 3 - (x - \tfrac{1}{2})^2 + \tfrac{1}{4} \\ &= \tfrac{13}{4} - (x - \tfrac{1}{2})^2 \end{aligned}$$

OCR AS Maths Quadratics 1 section test solutions

6. $2x^2+6x+1=2(x^2+3x)+1$

$$=2\left((x+\tfrac{3}{2})^2-\tfrac{9}{4}\right)+1$$
$$=2(x+\tfrac{3}{2})^2-\tfrac{9}{2}+1$$
$$=2(x+\tfrac{3}{2})^2-\tfrac{7}{2}$$

7. A quadratic graph with minimum point (1, -4) has the equation

$$y=(x-1)^2-4$$
$$=x^2-2x+1-4$$
$$=x^2-2x-3$$

8. A quadratic graph with maximum point (-2, 5) has the equation

$$y=5-(x+2)^2$$
$$=5-(x^2+4x+4)$$
$$=5-x^2-4x-4$$
$$=-x^2-4x+1$$

9. $y=x^2-2x-1$

$$=(x-1)^2-1-1$$
$$=(x-1)^2-2$$

The minimum point is (1, -2).

10. $y=-x^2+5x+2$

$$=2-(x^2-5x)$$
$$=2-\left((x-\tfrac{5}{2})^2-\tfrac{25}{4}\right)$$
$$=2-(x-\tfrac{5}{2})^2+\tfrac{25}{4}$$
$$=\tfrac{33}{4}-(x-\tfrac{5}{2})^2$$

The maximum point is $\left(\tfrac{5}{2},\tfrac{33}{4}\right)$

OCR AS Maths Quadratics 2 section test solutions

Solutions to section test

1. $2x^2 + 5x - 1 = 0$

   $a = 2, b = 5, c = -1$

   Discriminant $= b^2 - 4ac = 5^2 - 4 \times 2 \times -1 = 25 + 8 = 33$

2. The discriminant is positive, so the equation has two real roots. Since the discriminant is not a perfect square, the roots are irrational.
   So the equation has two irrational roots.

3. (i) $x^2 + 3x + 1 = 0$

   $a = 1, b = 3, c = 1$

   Discriminant $= b^2 - 4ac = 3^2 - 4 \times 1 \times 1 = 9 - 4 = 5$

   (ii) $2x^2 - 3x + 4 = 0$

   $a = 2, b = -3, c = 4$

   Discriminant $= b^2 - 4ac = (-3)^2 - 4 \times 2 \times 4 = 9 - 32 = -23$

   (iii) $3x^2 + x - 2 = 0$

   $a = 3, b = 1, c = -2$

   Discriminant $= b^2 - 4ac = 1^2 - 4 \times 3 \times -2 = 1 + 24 = 25$

   If the discriminant is negative, there are no real roots.
   So equation (ii) only has no real roots.

4. $x^2 + 2x - 5 = 0$

   $a = 1, \quad b = 2, \quad c = -5$

   Discriminant $= 2^2 - 4 \times 1 \times -5 = 4 + 20 = 24$

   As this is a positive number which is not a perfect square, the roots are irrational, and so the quadratic formula is needed.

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times -5}}{2 \times 1}$$

$$= \frac{-2 \pm \sqrt{24}}{2}$$

$$= \frac{-2 \pm \sqrt{4} \times \sqrt{6}}{2}$$

$$= \frac{-2 \pm 2\sqrt{6}}{2}$$

$$= -1 \pm \sqrt{6}$$

5. $2x^2 - 11x + 15 = 0$

   $a = 2, \quad b = -11, \quad c = 15$

   Discriminant $= (-11)^2 - 4 \times 2 \times 15 = 121 - 120 = 1$
   As this is a square number, the equation can be factorised.
   $2 \times 15 = 30$, so need two numbers which multiply together to give 30, and add together to give -11. These are -5 and -6.

   $2x^2 - 5x - 6x + 15 = 0$

   $x(2x-5) - 3(2x-5) = 0$

   $(x-3)(2x-5) = 0$

   $x = 3$ or $\frac{5}{2}$

6. $3x^2 - 2x + 4 = 0$

   $a = 3, b = -2, c = 4$

   $b^2 - 4ac = (-2)^2 - 4 \times 3 \times 4 = 4 - 48 = -44$
   The discriminant is negative so there are no real roots.

7. $2x^2 - 5x - 4 = 0$

   $a = 2, b = -5, c = -4$

   $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{5 \pm \sqrt{(-5)^2 - 4 \times 2 \times -4}}{4} = \frac{5 \pm \sqrt{57}}{4}$$

8. If the equation has equal roots, then the discriminant is zero.
   $x^2 + kx + 2k - 3 = 0$

   $a = 1, b = k, c = 2k - 3$

$b^2 - 4ac = 0$

$k^2 - 4 \times 1(2k - 3) = 0$

$k^2 - 8k + 12 = 0$

$(k-2)(k-6) = 0$

$k = 2$ or $k = 6$

9. Let $y = x^2$

Then $y^2 + 6y + 4 = 0$

$$y = \frac{-6 \pm \sqrt{6^2 - 4 \times 1 \times 4}}{4} = \frac{-6 \pm \sqrt{20}}{2} = -3 \pm \sqrt{5}$$

So $x^2 = -3 \pm \sqrt{5}$

Both $-3 + \sqrt{5}$ and $-3 - \sqrt{5}$ are negative, so their square roots are not real.
So there are no real roots.

10. When the object hits the ground, $h = 0$

$20t - 5t^2 + 1 = 0$

$5t^2 - 20t - 1 = 0$

$$t = \frac{20 \pm \sqrt{(-20)^2 - 4 \times 5 \times -1}}{2 \times 5} = \frac{20 \pm \sqrt{420}}{10}$$

$= 4.05$ or -0.05

As t represents time, it must be positive, so it hits the ground after 4.05 seconds.

OCR AS Maths Equations 1 section test solutions

Solutions to section test

1. $x+3y=5 \quad (1)$
$3x-y=5 \quad (2)\times 3$

$$\begin{aligned} x+3y&=5 \\ 9x-3y&=15 \\ \hline \text{Adding: } 10x \quad &=20 \\ x&=2 \end{aligned}$$

Substituting x = 2 into equation (1): $2+3y=5$
$$3y=3$$
$$y=1$$

2. $7a-3b=6 \quad (1)\times 2 \qquad 14a-6b=12$
$3a-2b=5 \quad (2)\times 3 \qquad 9a-6b=15$

Subtracting: $5a = \quad -3$
$$a=-\tfrac{3}{5}$$

Substituting $a=-\tfrac{3}{5}$ into equation (2): $-\tfrac{9}{5}-2b=5$
$$2b=-\tfrac{9}{5}-5=-\tfrac{34}{5}$$
$$b=-\tfrac{17}{5}$$

3. $5a+7b=17 \quad (1)$
$a=1-3b \quad (2)$

Substituting (2) into (1): $5(1-3b)+7b=17$
$$5-15b+7b=17$$
$$-8b=12$$
$$b=-\tfrac{3}{2}$$

Substituting $b=-\tfrac{3}{2}$ into (2): $a=1-3\times-\tfrac{3}{2}=1+\tfrac{9}{2}=\tfrac{11}{2}$

4. $2x=5y-2 \quad (1)$
$6y=1+4x \quad (2)$

Substituting (1) into (2): $6y=1+2(5y-2)$
$$6y=1+10y-4$$
$$-4y=-3$$
$$y=\tfrac{3}{4}$$

Substituting $y=\tfrac{3}{4}$ into (1): $2x=5\times\tfrac{3}{4}-2=\tfrac{15}{4}-2=\tfrac{7}{4}$
$$x=\tfrac{7}{8}$$

OCR AS Maths Equations 1 section test solutions

5. $s^2 + 2t^2 = 6 \quad (1)$

$3s - t = 5 \quad (2)$

$(2) \Rightarrow t = 3s - 5$

Substituting into (1): $s^2 + 2(3s-5)^2 = 6$

$$s^2 + 2(9s^2 - 30s + 25) = 6$$
$$s^2 + 18s^2 - 60s + 50 = 6$$
$$19s^2 - 60s + 44 = 0$$
$$(19s - 22)(s - 2) = 0$$
$$s = \tfrac{22}{19} \text{ or } s = 2$$

When $s = 2, t = 3 \times 2 - 5 = 1$

When $s = \frac{22}{19}, t = 3 \times \frac{22}{19} - 5 = \frac{66}{19} - 5 = -\frac{29}{19}$

So the values of t are t = 1 and $t = -\frac{29}{19}$

6. $x^2 - x - 1 = 2x + 3$

$x^2 - 3x - 4 = 0$

$(x+1)(x-4) = 0$

$x = -1$ and $x = 4$

7. $x^2 + 2y = 5 \quad (1)$

$2x - 3y = 12 \quad (2)$

$(2) \Rightarrow x = \dfrac{12+3y}{2}$

Substituting into (1): $\left(\dfrac{12+3y}{2}\right)^2 + 2y = 5$

$$\frac{144 + 72y + 9y^2}{4} + 2y = 5$$
$$144 + 72y + 9y^2 + 8y = 20$$
$$9y^2 + 80y + 124 = 0$$
$$(9y + 62)(y + 2) = 0$$
$$y = -\tfrac{62}{9} \text{ or } y = -2$$

8. $x^2 + 2xy + 1 = 0 \quad (1)$

$y + x = 1 \quad (2)$

$(2) \Rightarrow y = 1 - x$

Substituting into (1): $x^2+2x(1-x)+1=0$

$$x^2+2x-2x^2+1=0$$

$$-x^2+2x+1=0$$

$$x^2-2x-1=0$$

$a=1, b=-2, c=-1$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{2\pm\sqrt{4-4\times1\times-1}}{2}$$

$$=\frac{2\pm\sqrt{8}}{2}=\frac{2\pm2\sqrt{2}}{2}=1\pm\sqrt{2}$$

$y=1-x=1-\left(1\pm\sqrt{2}\right)=1-1\mp\sqrt{2}=\mp\sqrt{2}$

The solutions are $x=1+\sqrt{2}, y=-\sqrt{2}$ and $x=1-\sqrt{2}, y=\sqrt{2}$

9. $2x^2+y^2=21$ (1)

$y=2x-3$ (2)

Substituting (2) into (1): $2x^2+(2x-3)^2=21$

$$2x^2+4x^2-12x+9=21$$

$$6x^2-12x-12=0$$

$$x^2-2x-2=0$$

$a=1, b=-2, c=-2$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{2\pm\sqrt{4-4\times1\times-2}}{2}$$

$$=\frac{2\pm\sqrt{12}}{2}=\frac{2\pm2\sqrt{3}}{2}=1\pm\sqrt{3}$$

Substituting into (2): $x=1+\sqrt{3}\Rightarrow y=2\left(1+\sqrt{3}\right)-3=-1+2\sqrt{3}$

$$x=1-\sqrt{3}\Rightarrow y=2\left(1-\sqrt{3}\right)-3=-1-2\sqrt{3}$$

The solutions are $x=1+\sqrt{3}, y=-1+2\sqrt{3}$ and $x=1-\sqrt{3}, y=-1-2\sqrt{3}$

10. $x^2-4x+2=2x+k$

$x^2-6x+2-k=0$

There is a repeated root if the discriminant is zero.

$a=1, b=-6, c=2-k$

$b^2-4ac=0$

$(-6)^2-4\times1(2-k)=0$

$36-8+4k=0$

$4k=-28$

$k=-7$

Solutions to section test

1. $5x - 2 < 3x + 8$

   $2x - 2 < 8$

   $2x < 10$

   $x < 5$

2. $\dfrac{x+1}{2} \leq \dfrac{2-x}{3}$

   $3(x+1) \leq 2(2-x)$

   $3x + 3 \leq 4 - 2x$

   $5x + 3 \leq 4$

   $5x \leq 1$

   $x \leq \frac{1}{5}$

3. $2(1-2x) - x > 3(x+1) + 7$

   $2 - 4x - x > 3x + 3 + 7$

   $2 - 5x > 3x + 10$

   $2 > 8x + 10$

   $-8 > 8x$

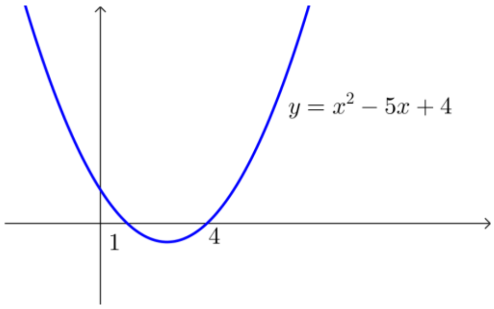
   $-1 > x$

   $x < -1$

4. $x^2 - 5x + 4 > 0$

   $(x-1)(x-4) > 0$

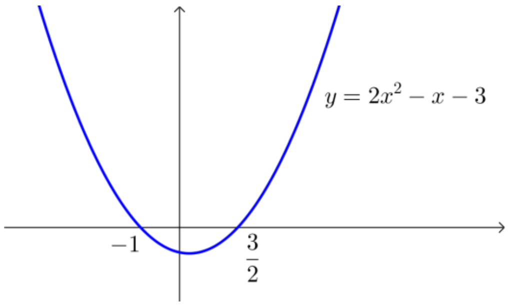
   From graph, $x < 1$ or $x > 4$



5. $2x^2 - x - 3 > 0$

   $(2x-3)(x+1) > 0$

   From graph, $x < -1$ or $x > \frac{3}{2}$



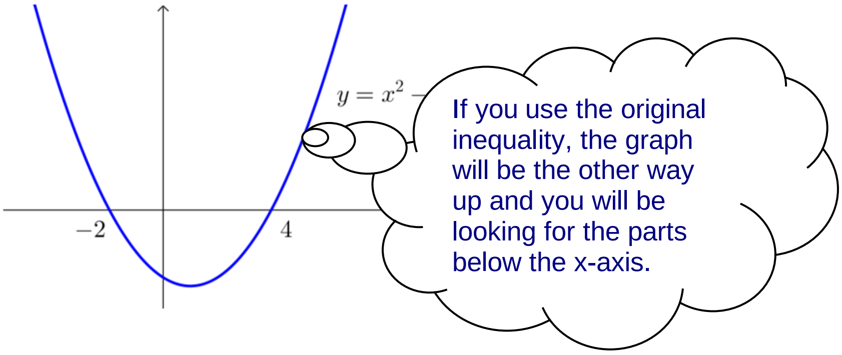
OCR AS Maths Equations & inequalities 2 Section test solns

6. $8+2x-x^2<0$

$x^2-2x-8>0$

$(x-4)(x+2)>0$

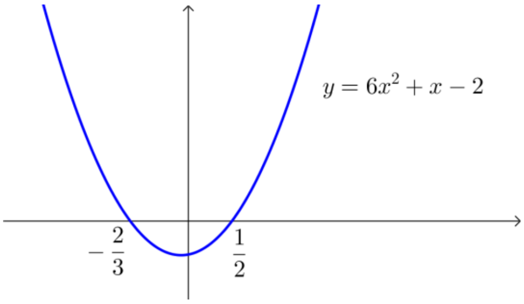
From graph, $x<-2$ or $x>4$.



7. $6x^2+x-2\geq 0$

$(3x+2)(2x-1)\geq 0$

From graph, $x\leq -\frac{2}{3}$ or $x\geq \frac{1}{2}$

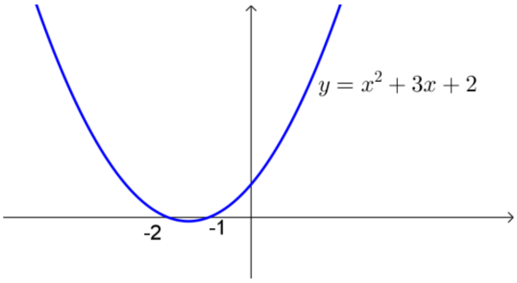


8. $2x^2-x+1<x^2-4x-1$

$x^2+3x+2<0$

$(x+1)(x+2)<0$

From graph, $-2<x<-1$.



9. If the equation has at least one real root, then discriminant ≥ 0.

$a=1, b=k-1, c=4$

$b^2-4ac\geq 0$

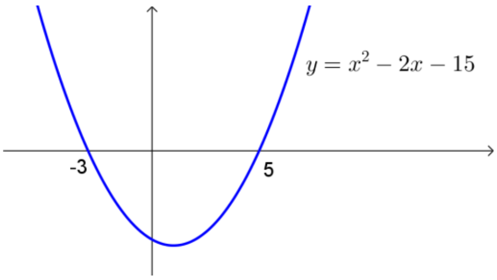
$(k-1)^2-4\times 1\times 4\geq 0$

$k^2-2k+1-16\geq 0$

$k^2-2k-15\geq 0$

$(k-5)(k+3)\geq 0$

From graph, $k\leq -3$ or $k\geq 5$



10. The boundary of the shaded region is dotted, so the line $y=2x+1$ is not included in the region.

The shaded area is above the line, so $y>2x+1$