Section 1: Polynomial functions and graphs

Exercise level 1

1. Given that $f(x) = x^3 + 2x^2 - 5x + 4$ and $g(x) = x^3 - 3x^2 + 1$, find
   (i) f(x) + g(x)
   (ii) f(x) – g(x)

2. Given that $p(x) = 2x^3 - 5x^2 + 3x - 2$ and $q(x) = x^3 - 2x^2 + 1$, find
   (i) q(x) – p(x)
   (ii) 2p(x) + 3q(x)

3. Given that $f(x) = x^3 + 5x^2 - 3$ and $g(x) = 3x^4 - 2x^3 + x$, find
   (i) g(x) – 3f(x)
   (ii) (2x + 1)f(x)

4. Expand the brackets and simplify the following as far as possible:
   (i) $(x-2)(2x^2 - 3x + 1)$
   (ii) $(3x-2)(x^3 - 2x + 4)$
   (iii) $(2x+1)(x^3 + 2x^2 - 3x - 5)$
   (iv) $(x+3)(2x-1)(x-4)$

5. Given $p(x) = 2x^2 + x - 1$ and $q(x) = 2x - 1$ find
   (i) $p(x) + q(x)$
   (ii) $p(x)q(x)$

6. Sketch the following graphs:
   (i) $y = (x+1)(x-3)(x+4)$
   (ii) $y = (x+2)^2(2x-1)$

Section 1: Polynomial functions and graphs

Exercise level 2

1. Expand the brackets and simplify the following as far as possible:
   (i) $(3x^2 - x + 2)(2x^2 + 5x - 1)$
   (ii) $(2x+3)(x-2)(x^2+1)$

2. Sketch the following graphs:
   (i) $y = x(3-x)(2x+3)$
   (ii) $y = x^2(x-2)(x+3)$
   (iii) $y = (x-2)^2(3x+4)^2$

3. Given that $\mathrm{f}(x) = x^2 + x + 1$ and $\mathrm{g}(x) = 2x^4 - x^3 + 2$ find
   (i) $[\mathrm{f}(x)]^2$
   (ii) $\mathrm{g}(x) - \mathrm{f}(x)$
   (iii) $\mathrm{f}(x)\mathrm{g}(x)$
   (iv) $\mathrm{f}(x)\left(\mathrm{g}(x) - \mathrm{f}(x)\right)$

4. Sketch a possible graph of $y = \mathrm{f}(x)$ where $\mathrm{f}(x)$ is a degree 4 polynomial and the equation $\mathrm{f}(x) = 0$ has
   (i) exactly 4 real roots
   (ii) exactly 1 real root and two local minimum points
   (iii) exactly 2 real roots and two local maximum points

Section 1: Polynomial functions and graphs

Exercise level 3 (Extension)

1. John is investigating a degree 5 polynomial, and its associated graph

$$y = \tfrac{1}{120}x^5 - \tfrac{1}{6}x^3 + x$$

(i) Sketch a degree 5 polynomial graph with positive coefficient of x^5 and with the greatest possible number of local maxima and minima.

(ii) John's first enquiry was to substitute (-*x*) for *x*. Deduce what John found out about the shape of the graph.

(iii) John's second step was to investigate the intercepts of the graph with the *x*-axis. Factorise the polynomial, and prove that there is only one real solution for the equation

$$\tfrac{1}{120}x^5 - \tfrac{1}{6}x^3 + x = 0$$

(iv) Next, John used a spreadsheet to further investigate the shape of the graph. He produced the following table:

x	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5
y	0.00	0.48	0.84	1.00	0.93	0.71	0.53	0.73	1.87	4.69

He used the table to make very rough estimates of the positions of any local minima and maxima. Write down approximations for the coordinates.

(v) From these coordinates, and his deductions in (ii) and (iii), John made a new sketch of the graph. Sketch the graph using your deductions.

(vi) Finally, John was excited to spot that the *x*-values of his intercept and maxima and minima seemed to be close to some of those of $y = \sin x$ where *x* is given in radians. Use a calculator (in 'radians' mode) to compare the values of the graph with the values of $y = \sin x$ when $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$, giving your answers to 3 decimal places. Sketch the two graphs on the same set of axes, and suggest for what values of *x* the polynomial would be a reasonable approximation for sin *x*.

2. Two researchers are separately carrying out the same experiment, and are each trying to find a polynomial graph to represent their results.

(i) Jane finds that her experiments yield the three data points A (2, 2), B (3, 2) and C (6, -2). She finds that all three data points fit on a quadratic graph which passes through the origin with equation

$$y = ax^2 + bx$$

Find the equation of her graph, and draw a sketch of it.

(ii) Samira also discovers the three data points A, B and C, but she has time to find one additional data point D (0 , 3). With the extra point, Samira is able to find the following cubic polynomial which is satisfied by all four data points

$$y = -\tfrac{1}{12}x^3 + \tfrac{7}{12}x^2 - \tfrac{4}{3}x + 3$$

Draw a sketch of Samira's graph.

(iii) In checking Jane's and Samira's results, Mary decides to 'subtract' Jane's polynomial from that of Samira, to obtain another cubic polynomial and graph. Find Mary's new polynomial and write down its roots. Sketch the graph of Mary's new polynomial.

3. **(For this question you should use a graphical calculator or graphing software.)**
Research into the history of a city suggests that its population over the last few centuries is given approximately in the following table:

Date	1700	1800	1900	2000
x	0	1	2	3
Population (millions) ***y***	1.0	1.5	2.5	4.5

These data could be modelled by using either a quadratic or a cubic polynomial graph, and my computer gives me approximations with equations

$$y = \tfrac{3}{8}x^2 + \tfrac{1}{40}x + \tfrac{41}{40}$$

$$y = \tfrac{5}{60}x^3 + \tfrac{5}{12}x + 1$$

(i) Use your calculator or computer to draw the original points (x, y) and to add the two approximations.

(ii) What does each approximate polynomial suggest was the population of the city in 1600 and 1500? Comment on the results.

(iii) In fact, the data are a very close fit to a non-polynomial graph

$$y = a(b + 2^x)$$

Find the values of a and b.

(iv) What does each of the three separate models predict as the population of the city in 2100?

Section 1: Quadratic graphs and equations

Exercise level 1

1. Factorise these quadratic expressions.

(i)	x^2+5x+6	(ii)	x^2+x-12	(iii)	x^2-9
(iv)	x^2-6x+8	(v)	$2x^2+3x+1$	(vi)	$3x^2+x-2$
(vii)	$4x^2-8x+3$	(viii)	$4x^2-25$	(ix)	$6x^2-x-12$

2. Factorise:

(i)	x^2-4x	(ii)	$x^2-17x-60$
(iii)	$x^2+4(x+1)$	(iv)	$3x^2-11x+6$

3. Solve these quadratic equations by factorising.

(i)	$x^2+4x+3=0$	(ii)	$x^2+5x-6=0$
(iii)	$x^2-6x+8=0$	(iv)	$x^2-7x-18=0$
(v)	$2x^2+5x+3=0$	(vi)	$2x^2+x-6=0$

4. Write down the equation of the line of symmetry and the coordinates of the vertex of each of the following quadratic graphs:

(i)	$y=(x-4)^2+1$	(ii)	$y=(x+2)^2-3$
(iii)	$y=(2x-1)^2-5$	(iv)	$y=3-(x+1)^2$

5. A quadratic graph has minimum point (-1, 2). Find an equation for the graph.

6. A quadratic graph has maximum point (2, 5). Find an equation for the graph.

7. Write each of the following quadratic functions in completed square form:

(i)	x^2+2x-3	(ii)	x^2-6x+1
(iii)	x^2+x+1	(iv)	$-x^2+5x$
(v)	$2x^2+4x+3$	(vi)	$3x^2+8x-2$

8. Using your answers for each of the quadratic functions in question 7, write down the coordinates of the minimum or maximum point (the vertex) of the graph.

(i)	$y=x^2+2x-3$	(ii)	$y=x^2-6x+1$
(iii)	$y=x^2+x+1$	(iv)	$y=-x^2+5x$
(v)	$y=2x^2+4x+3$	(vi)	$y=3x^2+8x-2$

Section 1: Quadratic graphs and equations

Exercise level 2

1. Factorise:
   (i) $ax^2 - 2ax - 3a$ (ii) $2cx^2 + c(6a+b)x + 3abc$

2. Simplify these expressions where possible.
   (i) $\dfrac{x^2+x-6}{x^2-x-2}$ (ii) $\dfrac{x^2-4x+4}{x^2+x-6}$
   (iii) $\dfrac{x^2+x-2}{x^2+4x+3}$ (iv) $\dfrac{4x^2-1}{4x^2-4x-3}$
   (v) $\dfrac{2x+3}{3x+1}\times\left(3x^2-2x-1\right)$ (vi) $\dfrac{x+2}{2x^2-x-1}\div\dfrac{x^2-x-6}{2x+1}$

3. Solve these quadratic equations by factorising.
   (i) $4x^2 - 3x - 10 = 0$ (ii) $6x^2 - 19x + 10 = 0$

4. The length of a rectangle is 3 cm greater than its width. The area of the rectangle is 40 cm². Find the length and width of the rectangle.

5. Solve the following equations.
   (i) $x^4 - 5x^2 + 4 = 0$ (ii) $4x^4 + 11x^2 - 3 = 0$

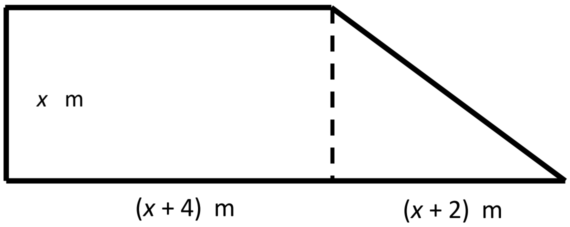
6. (i) Write $x^2 + 4x + 1$ in the completed square form.
   (ii) Hence write down the equation of the line of symmetry and the coordinates of the vertex of the graph $y = x^2 + 4x + 1$.
   (iii) Sketch the graph.

7. (i) Write $x^2 - 3x + 1$ in the completed square form.
   (ii) Hence write down the equation of the line of symmetry and the coordinates of the vertex of the graph $y = x^2 - 3x + 1$.
   (iii) Sketch the graph.

Section 1: Quadratic graphs and equations

Exercise level 3 (Extension)

1. The garden shown in the diagram is in the shape of a rectangle, with an attached triangular area. It is surrounded by a continuous fence, and the dimensions shown are measured in metres.



   (i) Write down a formula for the area A m^2 of the garden.
   (ii) Find a second formula for the length P of the fence in metres, leaving a square root in your formula.
   (iii) The area of the garden is 200 m^2. Find the length of the fence, giving your answer to 3 significant figures.

2. In a cinema, there are n rows of seats set out in a rectangular block of 300. After the cinema is enlarged, half as many rows plus two more are added, though with 5 fewer seats in each new row. This gives the cinema 120 extra seats. How many seats are in each of the old and new rows?

3. A rectangular enclosure is to be constructed against a long straight wall. The enclosure is to be built using 100 metres of fencing.
   (i) Let the width of the enclosure be x metres. Draw a sketch, and write down a formula for the area of the enclosure.
   (ii) By completing the square, find the maximum possible area that can be enclosed, and find the dimensions of the enclosure to give that maximum area.
   (iii) Sketch the graph of your formula for the area. Interpret and explain the values of the intercepts of the graph with the x-axis in terms of the shape of the enclosure.

Section 2: The quadratic formula

Exercise level 1

1. Without solving the equation, state how many solutions there are for each of the following quadratic equations:

(i)	$3x^2+2x+5=0$	(ii)	$2x^2-3x-2=0$
(iii)	$5x^2-6=0$	(iv)	$4x^2-8x+4=0$
(v)	$x^2-3x+3=0$	(vi)	$-5x^2-8x-10$

2. Use the quadratic formula to solve these equations. Give your answers in exact form.

(i)	$x^2+4x+1=0$	(ii)	$x^2-3x-1=0$
(iii)	$2x^2+2x-3=0$	(iv)	$3x^2-4x-2=0$

Section 2: The quadratic formula

Exercise level 2

1. Solve the following quadratic equations, where possible. Give answers in exact form.

   (i) $x^2+2x-2=0$ (ii) $x^2-3x+5=0$

   (iii) $2x^2+x-4=0$ (iv) $2x^2-5x-12=0$

   (v) $x^2-5x-3=0$ (vi) $3x^2+x+1=0$

   (vii) $4x^2+12x+9=0$ (viii) $4x^2+10x+5=0$

2. Solve the following equations, giving your answers in exact form.

   (i) $x=\frac{3}{x}-1$ (ii) $6\sqrt{x}-7=x$

3. A cylinder has height 20 cm and surface area 300 cm². Find the radius of the cylinder, to 3 s.f.

4. The equation $x^2+(3k+1)x+4k+13=0$ has a repeated root. Find the possible values of k.

5. What is the greatest possible value of k if the equation $2x^2-5x+k=0$ has at least one real root?

Section 2: The quadratic formula

Exercise level 3 (Extension)

1. When a stone is thrown upwards over the edge of a cliff, its height *h* metres above the point where it was thrown after *t* seconds is given by the formula

$$h = 20t - 5t^2$$

How many seconds after it is thrown does the stone pass the cliff edge on the way downwards? How long after it is thrown will the stone hit the sea which is 50 metres below the clifftop? (Give your answer to 3 significant figures.) How could you interpret the other solution of your quadratic equation?

2. (i) Write the expression

$$x^2 + 8x + c$$

in completed square form.

(ii) The equation

$$x^2 + 8x + c = 0$$

has real roots. Using the completed square format, find a condition in the form of an inequality for *c*.

(iii) How must this condition on *c* be amended so that the equation above in part (ii) has real unequal roots?

(iv) Sketch two graphs showing an example of two real equal roots in (ii) and two real unequal roots in (iii) above.

(v) By considering your sketch graphs, explain why there is no value of *c* which gives two real positive roots.

3. A rectangular car park has a perimeter of 184 metres, and the diagonal of the car park measures 68 metres.

(i) By labelling the length of the car park as *x* metres, formulate an equation and check that $x = 32$ satisfies the equation. Hence find the dimensions of the car park.

(ii) Sketch the graph of the quadratic expression in part (i), and interpret each intersection with the *x*-axis in terms of the car park.

Section 1: Simultaneous equations

Exercise level 1

1. Solve the following simultaneous equations:

(i) $2x+5y=11$
$2x-y=5$

(ii) $x+2y=6$
$4x+3y=4$

(iii) $3a-2b=4$
$5a+4b=3$

(iv) $2p-5q=5$
$3p-2q=-9$

(v) $5x+3y=9$
$y=3x-4$

(vi) $3a+2b=1$
$9a-4b=4$

2. Solve the following simultaneous equations:

(i) $x-y=-1$
$3x+2y=7$

(ii) $2x+y=0$
$x-3y=7$

(iii) $y-5x=-8$
$x+3y=0$

(iv) $x=2y-1$
$-x+3y=-1$

(v) $2p-4q=14$
$-p+3q=-5$

(vi) $3u-2v=-26$
$-u+6v=46$

Section 1: Simultaneous equations

Exercise level 2

1. Solve the following simultaneous equations.

   (i) $7x^2 + y^2 = 64$, $x + y = 4$

   (ii) $3x^2 - 2y^2 = -5$, $y - x = 1$

   (iii) $p^2 + pq = 2$, $q - p = 3$

   (iv) $8a^2 - b^2 = 2$, $2a + b = 1$

2. Solve the following simultaneous equations.

   (i) $x + y = 9$, $x^2 - 3xy + 2y^2 = 0$

   (ii) $xy = 8$, $3x - y = 10$

   (iii) $y = 4x$, $3y^2 - 2xy = 160$

3. In each of the following questions, find where the two graphs cross, and show the crossing points on a sketch.

   (i) $y = 3x - 2$, $y = x^2 - 3x - 9$

   (ii) $y + 2x = 3$, $y = 6 + 4x - x^2$

4. (i) By completing the square, find the coordinates of the vertex of the graph $y = x^2 + x + 1$.

   (ii) By putting the two expressions equal to each other in a single equation, find where the two graphs below cross:

   $$y = x^2 + x + 1$$
   $$y = 5x - 3$$

   (iii) Interpret your result by sketching the graphs.

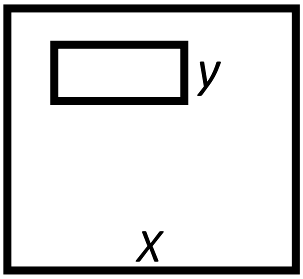
5. The line $y = 2x - 3$ touches the curve $y = x^2 + kx + 6$.
   Find the possible values of k.

Section 1: Simultaneous equations

Exercise level 3

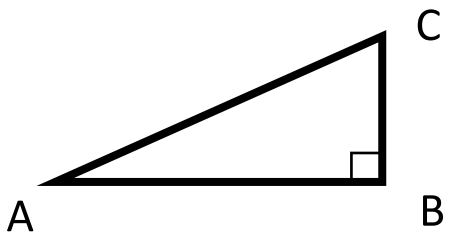
1. The diagram shows a plan of a square lawn, with a rectangular flowerbed cut from it. The flowerbed is half the length of the lawn. The lawn is x m long, and the flowerbed is y m in width.

   I have 100 m of plastic 'lawn edging' which I intend to use on all 4 sides of the lawn, and also all 4 sides of the flowerbed. I also have a packet of grass seed, which states that it will cover 279 m^2. What should be the dimensions of each of the lawn and flowerbed?



2. For centuries, people have used the properties of right-angled triangles to set out building works accurately.

   In the diagram, a 40 m length of rope is used to set out a right-angled triangle ABC. The length of rope AC is h m, and the length of rope AB is 2 m shorter than AC. The length of the remaining part of the rope BC is x m. The area of the triangle ABC is 60 m^2. Find the possible dimensions of the triangle.



3. A fishtank is 30 cm deep, and is formed from a cuboid with horizontal dimensions x cm by y cm. Find formulae for the surface area (the tank has no lid) and the volume of the tank.
   If the surface area is 6300 cm^2 and the volume is 45000 cm^3, find the size of the tank.

Section 2: Inequalities

Exercise level 1

1. Solve the following linear inequalities.

(i) $2x+3<10$	(ii) $5x+3\geq 2x-9$
(iii) $3x-1>7-x$	(iv) $4x+1\leq 6x-7$
(v) $5x+2>-7$	(vi) $3x-11\leq 5+4x$
(vii) $3(2-3x)\geq 5x+1$	(viii) $\frac{1}{3}(7+6x)<2-x$

2. (i) Write $x^2-11x+24$ in factorised form.

   (ii) Sketch the graph of $y=x^2-11x+24$, labelling the values of x where the graph crosses the x-axis.

   (iii) Use your graph to write down the solution of

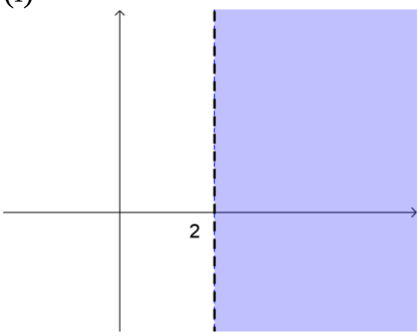
   $$x^2-11x+24\geq 0$$

3. Solve the following quadratic inequalities.

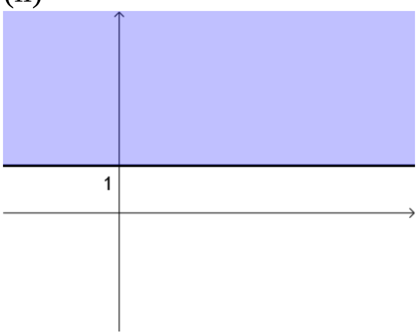
(i) $x^2-4x-12\leq 0$	(ii) $x^2-7x+6>0$
(iii) $x^2+2x-15\geq 0$	(iv) $2x^2-5x-3\leq 0$
(v) $3x^2+5x+2<0$	(vi) $4x^2-4x-3>0$

4. Write down an inequality to describe the shaded area in each of these diagrams.

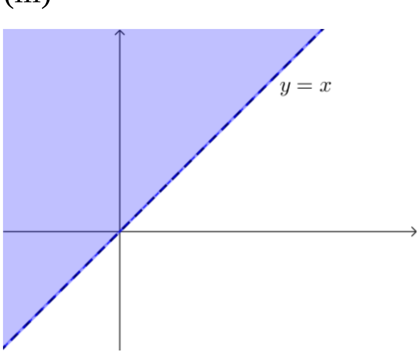
(i)



(ii)



(iii)



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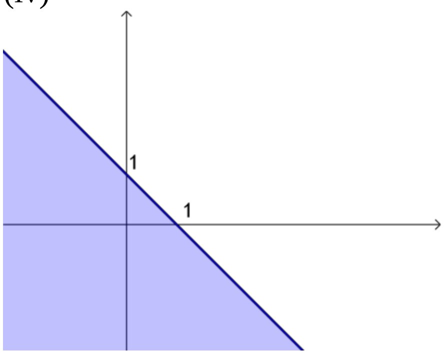


Section 2: Inequalities

Exercise level 2

1. Solve the following linear inequalities.
   (i) $5(x-3) \le 2(2x+3)$ (ii) $2(1-x) > 3x+4$
   (iii) $4(2x+5) \ge 3(3x-1)$ (iv) $\dfrac{2x+1}{3} > \dfrac{x-4}{2}$
   (v) $-\frac{1}{2}(4+3x) \ge 2x-1$ (vi) $\dfrac{x-1}{3} > -\dfrac{3-x}{2}$

2. In each of the following parts, use a sketch of appropriate quadratic graphs to solve the quadratic inequalities, and indicate on the sketch the values of x which represent the solution.
   (i) $x^2 - 5x + 6 < 0$
   (ii) $-2x^2 + x + 3 \ge 0$
   (iii) $x^2 + 8 < 2x^2 + x + 6$

3. Solve the following quadratic inequalities.
   (i) $1 - x - 2x^2 \ge 0$ (ii) $x^2 + 2x - 1 < 0$
   (iii) $x^2 \ge 3x + 10$ (iv) $x(x+3) > x + 8$

4. Show the regions represented by the following inequalities on graphs.
   (i) $y > x - 2$ (ii) $y \le 2x - 3$
   (iii) $y \ge x^2 + 1$ (iv) $y < x^2 + 2x - 3$

5. Find the set of values of k for which each of the quadratic equations below have no real roots.
   (i) $x^2 - 5x + k = 0$
   (ii) $x^2 + kx + k + 3 = 0$

Section 2: Inequalities

Exercise level 3 (Extension)

1. John is less than half his mother's age, though the sum of their ages is greater than sixty. John was born on his mother's twenty-sixth birthday.
   By expressing these facts as two inequalities and one equation, find the range of possible ages for John.

2. A room in a new hotel is designed so that its length x metres is 3 metres greater than its width. Because of the budget for carpet, the area is to be no more than 88 square metres, while for display purposes the perimeter of the room must be at least 30 metres.
   Write down two inequalities, and find the possible values for x.

3. A work of art includes a large cone. The volume of the cone is to be at most 25 cubic metres, and the length of the slant height of the cone is to be less than twice the radius. Write down two inequalities, and find the maximum height of the cone, to 3 significant figures.

OCR AS Mathematics Polynomials

Section 1: Polynomial functions and graphs

Solutions to Exercise level 1

1. (i) $f(x)+g(x)=\left(x^3+2x^2-5x+4\right)+\left(x^3-3x^2+1\right)$
$=x^3+2x^2-5x+4+x^3-3x^2+1$
$=x^3+x^3+2x^2-3x^2-5x+4+1$
$=2x^3-x^2-5x+5$

(ii) $f(x)-g(x)=\left(x^3+2x^2-5x+4\right)-\left(x^3-3x^2+1\right)$
$=x^3+2x^2-5x+4-x^3+3x^2-1$
$=x^3-x^3+2x^2+3x^2-5x+4-1$
$=5x^2-5x+3$

2. (i) $q(x)-p(x)=\left(x^3-2x^2+1\right)-\left(2x^3-5x^2+3x-2\right)$
$=x^3-2x^2+1-2x^3+5x^2-3x+2$
$=x^3-2x^3-2x^2+5x^2-3x+1+2$
$=-x^3+3x^2-3x+3$

(ii) $2p(x)+3q(x)=2\left(2x^3-5x^2+3x-2\right)+3\left(x^3-2x^2+1\right)$
$=4x^3-10x^2+6x-4+3x^3-6x^2+3$
$=4x^3+3x^3-10x^2-6x^2+6x-4+3$
$=7x^3-16x^2+6x-1$

3. (i) $g(x)-3f(x)=3x^4-2x^3+x-3\left(x^3+5x^2-3\right)$
$=3x^4-2x^3+x-3x^3-15x^2+9$
$=3x^4-2x^3-3x^3-15x^2+x+9$
$=3x^4-5x^3-15x^2+x+9$

(ii) $(2x+1)f(x)=(2x+1)\left(x^3+5x^2-3\right)$
$=2x\left(x^3+5x^2-3\right)+\left(x^3+5x^2-3\right)$
$=2x^4+10x^3-6x+x^3+5x^2-3$
$=2x^4+10x^3+x^3+5x^2-6x-3$
$=2x^4+11x^3+5x^2-6x-3$

OCR AS Maths Polynomials 1 Exercise solutions

4. (i) $(x-2)(2x^2-3x+1)=x(2x^2-3x+1)-2(2x^2-3x+1)$
$=2x^3-3x^2+x-4x^2+6x-2$
$=2x^3-7x^2+7x-2$

(ii) $(3x-2)(x^3-2x+4)=3x(x^3-2x+4)-2(x^3-2x+4)$
$=3x^4-6x^2+12x-2x^3+4x-8$
$=3x^4-2x^3-6x^2+16x-8$

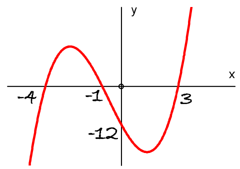
(iii) $(2x+1)(x^3+2x^2-3x-5)$
$=2x(x^3+2x^2-3x-5)+(x^3+2x^2-3x-5)$
$=2x^4+4x^3-6x^2-10x+x^3+2x^2-3x-5$
$=2x^4+5x^3-4x^2-13x-5$

(iv) $(x+3)(2x-1)(x-4)=(x+3)(2x^2-8x-x+4)$
$=(x+3)(2x^2-9x+4)$
$=x(2x^2-9x+4)+3(2x^2-9x+4)$
$=2x^3-9x^2+4x+6x^2-27x+12$
$=2x^3-3x^2-23x+12$

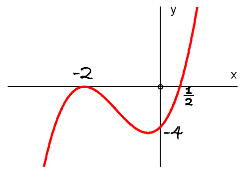
5. (i) $p(x)+q(x)=(2x^2+x-1)+(2x-1)$
$=2x^2+3x-2$

(ii) $p(x)q(x)=(2x^2+x-1)(2x-1)$
$=4x^3-3x+1$

6. (i) $y=(x+1)(x-3)(x+4)$
This is a cubic graph which cuts the x-axis at (-1, 0), (3, 0) and (-4, 0).
When $x=0$, $y=1\times-3\times4=-12$
When x is large and positive, y is positive.
When x is large and negative, y is negative.



(ii) $y=(x+2)^2(2x-1)$
This is a cubic graph which touches the x-axis at (-2, 0) and cuts the x-axis at $(\frac{1}{2},0)$.
When $x=0$, $y=2^2\times-1=-4$
When x is large and positive, y is positive.
When x is large and negative, y is negative.



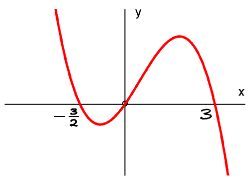
Section 1: Polynomial functions and graphs

Solutions to Exercise level 2

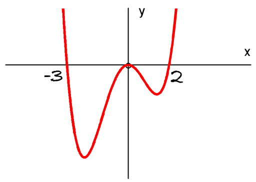
1. (i) $(3x^2 - x + 2)(2x^2 + 5x - 1)$
$= 3x^2(2x^2 + 5x - 1) - x(2x^2 + 5x - 1) + 2(2x^2 + 5x - 1)$
$= 6x^4 + 15x^3 - 3x^2 - 2x^3 - 5x^2 + x + 4x^2 + 10x - 2$
$= 6x^4 + 13x^3 - 4x^2 + 11x - 2$

(ii) $(2x+3)(x-2)(x^2+1) = (2x+3)(x^3 - 2x^2 + x - 2)$
$= 2x(x^3 - 2x^2 + x - 2) + 3(x^3 - 2x^2 + x - 2)$
$= 2x^4 - 4x^3 + 2x^2 - 4x + 3x^3 - 6x^2 + 3x - 6$
$= 2x^4 - x^3 - 4x^2 - x - 6$

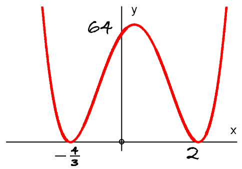
2. (i) $y = x(3-x)(2x+3)$
This is a cubic graph which cuts the x-axis at (0, 0), (3, 0) and $(-\frac{3}{2}, 0)$.
When $x = 0$, $y = 0$
When x is large and positive, y is negative.
When x is large and negative, y is positive.



(ii) $y = x^2(x-2)(x+3)$
This is a quartic graph which touches the x-axis at (0, 0) and cuts the x-axis at (2, 0) and (-3, 0).
When $x = 0$, $y = 0$
When x is large and positive, y is positive.
When x is large and negative, y is positive.



(iii) $y = (x-2)^2(3x+4)^2$
This is a quartic graph which touches the x-axis at (2, 0) and $(-\frac{4}{3}, 0)$.
When $x = 0$, $y = 2^2 \times 4^2 = 64$
When x is large and positive, y is positive.
When x is large and negative, y is positive.



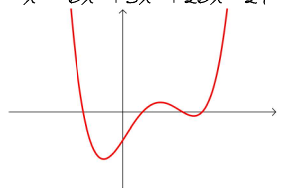
3. (i) $[f(x)]^2 = (x^2 + x + 1)(x^2 + x + 1)$
$= x^4 + 2x^3 + 3x^2 + 2x + 1$

(ii) $g(x) - f(x) = 2x^4 - x^3 + 2 - (x^2 + x + 1)$
$= 2x^4 - x^3 + 2 - x^2 - x - 1$
$= 2x^4 - x^3 - x^2 - x + 1$

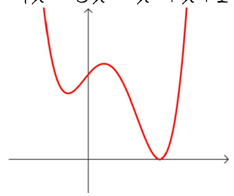
(iii) $f(x)g(x) = (x^2 + x + 1)(2x^4 - x^3 + 2)$
$= 2x^6 + x^5 + x^4 - x^3 + 2x^2 + 2x + 2$

(iv) $f(x)(g(x) - f(x)) = f(x)g(x) - [f(x)]^2$
$= (2x^6 + x^5 + x^4 - x^3 + 2x^2 + 2x + 2)$
$- (x^4 + 2x^3 + 3x^2 + 2x + 1)$
$= 2x^6 + x^5 - 3x^3 - x^2 + 1$

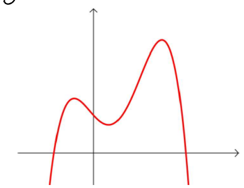
4. (i) e.g. $y = (x+2)(x-1)(x-3)(x-4)$
$= x^4 - 6x^3 + 3x^2 + 26x - 24$



(ii) e.g. $y = (x-1)^2(4x^2 + 3x + 1)$
$= 4x^4 - 5x^3 - x^2 + x + 1$



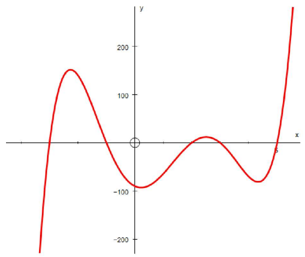
(iii) e.g. $y = -8x^4 + 10x^3 + 2x^2 - 2x - 1$



Section 1: Polynomial functions and graphs

Solutions to Exercise level 3 (Extension)

1. (i)



e.g. $y = (x+3)(x+1)(x-2)(x-3)(x-5)$ or many others with 5 or fewer points where it crosses the x-axis, and 4 local maxima/minima.

(ii) Putting $-x$ for x in $y = \frac{1}{120}x^5 - \frac{1}{6}x^3 + x$

gives $y = \frac{1}{120}(-x)^5 - \frac{1}{6}(-x)^3 + (-x)$

$= -\left(\frac{1}{120}x^5 - \frac{1}{6}x^3 + x\right)$

so the graph has half-turn symmetry about O (called an odd graph).

(iii) $\frac{1}{120}x^5 - \frac{1}{6}x^3 + x = 0$

$\Rightarrow \frac{1}{120}x(x^4 - 20x^2 + 120) = 0$

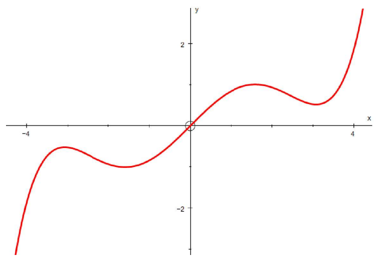
and for the quadratic expression in x^2, discriminant $= 20^2 - 4 \times 1 \times 120$

$= -80$

so there are no other intercepts other than $x = 0$.

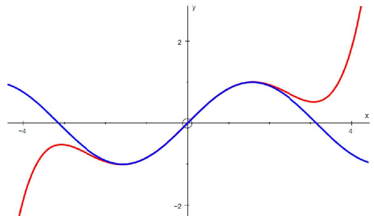
(iv) There is a maximum near (1.5, 1) and a minimum near (3, 0.53), so from part (ii) there is a minimum near (-1.5, -1) and a maximum near (-3, -0.53).

(v)



(vi)

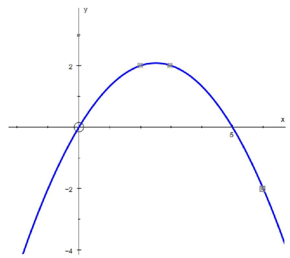
x	0	π/4	π/2	3π/4	π
y = sinx	0	0.707	1	0.707	0
y = f(x)	0	0.707	1.005	0.781	0.524



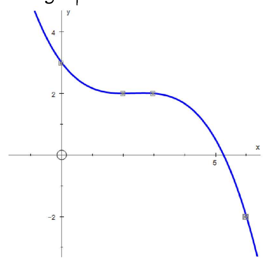
The polynomial could be a useful approximation for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

2. (i) point A gives $2 = 4a + 2b$
point B gives $2 = 9a + 3b$ $\Big\} \Rightarrow a = -\frac{1}{3},\ b = \frac{5}{3}$

so Jane's graph is $y = -\frac{1}{3}x^2 + \frac{5}{3}x$

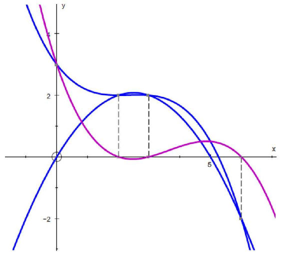


(ii) Samira's graph

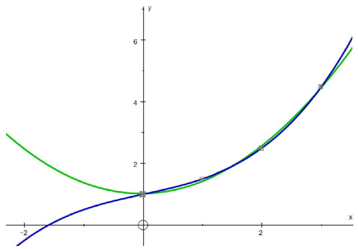


(iii) Mary's new polynomial is $y = -\frac{1}{12}x^3 + \frac{11}{12}x^2 - 3x + 3$

Since Jane's and Samira's graphs both pass through A, B and C, Mary's cubic polynomial has roots at $x = 2, 3$, and 6.



3. (i)



(ii) The estimates are:

date	1600	1500
quadratic	1.375 m	2.475 m
cubic	0.5 m	-0.5 m

Both models give absurd estimates for 1500! And greatly different for 1600.

(iii) If $y = a(b + 2^x)$,

then $x = 0, y = 1 \Rightarrow 1 = a(b+1)$

and $x = 1, y = 1.5 \Rightarrow 1.5 = a(b+2)$

Dividing: $\frac{b+2}{b+1} = 1.5$

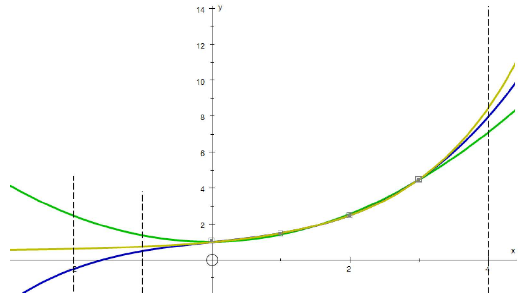
$\Rightarrow b = 1, a = \frac{1}{2}$

so $y = \frac{1}{2}(1 + 2^x)$ (and the other points fit exactly)

(iv) The predictions for year 2100 are:
quadratic polynomial gives 7.125 million
cubic polynomial gives 8.0 million
exponential function gives 8.5 million

For completeness, the calculations for all three models are below:

date	1500	1600	1700	1800	1900	2000	2100
x	-2	-1	0	1	2	3	4
y	??	??	1.0	1.5	2.5	4.5	??
quadratic	2.475	1.375	1.025	1.425	2.575	4.475	7.125
cubic	-0.5	0.5	1.0	1.5	2.5	4.5	8.0
exponential	0.625	0.75	1.0	1.5	2.5	4.5	8.5



Section 1: Quadratic graphs and equations

Solutions to Exercise level 1

1. (i) $x^2+5x+6 = x^2+3x+2x+6$
$= x(x+3)+2(x+3)$
$= (x+2)(x+3)$

(ii) $x^2+x-12 = x^2+4x-3x-12$
$= x(x+4)-3(x+4)$
$= (x-3)(x+4)$

(iii) $x^2-9 = (x+3)(x-3)$

(iv) $x^2-6x+8 = x^2-2x-4x+8$
$= x(x-2)-4(x-2)$
$= (x-4)(x-2)$

(v) $2x^2+3x+1 = 2x^2+x+2x+1$
$= x(2x+1)+1(2x+1)$
$= (x+1)(2x+1)$

(vi) $3x^2+x-2 = 3x^2+3x-2x-2$
$= 3x(x+1)-2(x+1)$
$= (3x-2)(x+1)$

(vii) $4x^2-8x+3 = 4x^2-2x-6x+3$
$= 2x(2x-1)-3(2x-1)$
$= (2x-3)(2x-1)$

(viii) $4x^2-25 = (2x+5)(2x-5)$

(ix) $6x^2-x-12 = 6x^2+8x-9x-12$
$= 2x(3x+4)-3(3x+4)$
$= (2x-3)(3x+4)$

2. (i) $x^2-4x = x(x-4)$

(ii) $x^2-17x-60 = (x-20)(x+3)$

(iii) $x^2+4(x+1) = x^2+4x+4$
$= (x+2)^2$

(iv) $3x^2-11x+6 = (3x-2)(x-3)$

3. (i) $x^2+4x+3=0$
$(x+3)(x+1)=0$
$x=-3$ or $x=-1$

(ii) $x^2+5x-6=0$
$(x+6)(x-1)=0$
$x=-6$ or $x=1$

(iii) $x^2-6x+8=0$
$(x-2)(x-4)=0$
$x=2$ or $x=4$

(iv) $x^2-7x-18=0$
$(x-9)(x+2)=0$
$x=9$ or $x=-2$

(v) $2x^2+5x+3=0$
$(2x+3)(x+1)=0$
$x=-\frac{3}{2}$ or $x=-1$

(vi) $2x^2+x-6=0$
$(2x-3)(x+2)=0$
$x=\frac{3}{2}$ or $x=-2$

4. (i) Line of symmetry is $x = 4$
Vertex (minimum point) is $(4, 1)$

(ii) Line of symmetry is $x = -2$
Vertex (minimum point) is $(-2, -3)$

(iii) Line of symmetry is $x = \frac{1}{2}$
Vertex (minimum point) is $(\frac{1}{2}, -5)$

(iv) Line of symmetry is $x = -1$
Vertex (maximum point) is (-1, 3).

5. Minimum point is (-1, 2)
Equation of graph is $y = (x+1)^2 + 2$
$= x^2 + 2x + 1 + 2$
$= x^2 + 2x + 3$

6. Maximum point is (2, 5)
Equation of graph is $y = 5 - (x-2)^2$
$= 5 - (x^2 - 4x + 4)$
$= 5 - x^2 + 4x - 4$
$= -x^2 + 4x + 1$

7. (i) $x^2 + 2x - 3 = (x+1)^2 - 1^2 - 3$
$= (x+1)^2 - 4$

(ii) $x^2 - 6x + 1 = (x-3)^2 - 3^2 + 1$
$= (x-3)^2 - 8$

(iii) $x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1$
$= \left(x + \frac{1}{2}\right)^2 - \frac{1}{4} + 1$
$= \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$

(iv) $-x^2 + 5x = -\left(x^2 - 5x\right)$
$= -\left(\left(x - \frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2\right)$
$= -\left(x - \frac{5}{2}\right)^2 + \frac{25}{4}$

(v) $2x^2 + 4x + 3 = 2\left(x^2 + 2x\right) + 3$
$= 2\left((x+1)^2 - 1^2\right) + 3$
$= 2(x+1)^2 - 2 + 3$
$= 2(x+1)^2 + 1$

(vi) $3x^2 + 8x - 2 = 3\left(x^2 + \frac{8}{3}x\right) - 2$
$= 3\left(\left(x + \frac{4}{3}\right)^2 - \left(\frac{4}{3}\right)^2\right) - 2$
$= 3\left(x + \frac{4}{3}\right)^2 - 3 \times \frac{16}{9} - 2$
$= 3\left(x + \frac{4}{3}\right)^2 - \frac{16}{3} - 2$
$= 3\left(x + \frac{4}{3}\right)^2 - \frac{22}{3}$

8. (i) $(-1, -4)$ minimum

(ii) $(3, -8)$ minimum

(iii) $\left(-\frac{1}{2}, \frac{3}{4}\right)$ minimum

(iv) $\left(\frac{5}{2}, \frac{25}{4}\right)$ maximum

(v) $(-1, 1)$ minimum

(vi) $\left(-\frac{4}{3}, -\frac{22}{3}\right)$ minimum

Section 1: Quadratic graphs and equations

Solutions to Exercise level 2

1. (i) $ax^2 - 2ax - 3a = a(x^2 - 2x - 3)$
$= a(x-3)(x+1)$

(ii) $2cx^2 + c(6a+b)x + 3abc = c\left[2x^2 + (6a+b)x + 3ab\right]$
$= c(x+3a)(2x+b)$

2. (i) $\dfrac{x^2+x-6}{x^2-x-2} = \dfrac{(x+3)\cancel{(x-2)}}{\cancel{(x-2)}(x+1)} = \dfrac{x+3}{x+1}$

(ii) $\dfrac{x^2-4x+4}{x^2+x-6} = \dfrac{(x-2)^{\cancel{2}}}{(x+3)\cancel{(x-2)}} = \dfrac{x-2}{x+3}$

(iii) $\dfrac{x^2+x-2}{x^2+4x+3} = \dfrac{(x+2)(x-1)}{(x+3)(x+1)}$
This expression cannot be simplified.

(iv) $\dfrac{4x^2-1}{4x^2-4x-3} = \dfrac{\cancel{(2x+1)}(2x-1)}{\cancel{(2x+1)}(2x-3)} = \dfrac{2x-1}{2x-3}$

(v) $\dfrac{2x+3}{3x+1}\times(3x^2-2x-1) = \dfrac{2x+3}{\cancel{3x+1}}\times\cancel{(3x+1)}(x-1) = (2x+3)(x-1)$

(vi) $\dfrac{x+2}{2x^2-x-1} \div \dfrac{x^2-x-6}{2x+1} = \dfrac{x+2}{(2x+1)(x-1)} \div \dfrac{(x-3)(x+2)}{2x+1}$
$= \dfrac{\cancel{x+2}}{\cancel{(2x+1)}(x-1)} \times \dfrac{\cancel{2x+1}}{(x-3)\cancel{(x+2)}}$
$= \dfrac{1}{(x-1)(x-3)}$

3. (i) $4x^2 - 3x - 10 = 0$
$(4x+5)(x-2) = 0$
$x = -\frac{5}{4}$ or $x = 2$

(ii) $6x^2 - 19x + 10 = 0$
$(3x-2)(2x-5) = 0$
$x = \frac{2}{3}$ or $x = \frac{5}{2}$

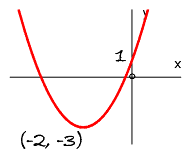
4. Let x be the width of the rectangle, so the length is x + 3.
Area $= x(x+3)$
$x(x+3) = 40$
$x^2 + 3x = 40$
$x^2 + 3x - 40 = 0$
$(x+8)(x-5) = 0$
$x = -8$ or 5
Dimensions must be positive, so width is 5 cm and length is 8 cm.

5. (i) Let $y = x^2$
$y^2 - 5y + 4 = 0$
$(y-1)(y-4) = 0$
$y = 1$ or 4
$x^2 = 1$ or 4
$x = \pm 1$ or ± 2

(ii) Let $y = x^2$
$4y^2 + 11y - 3 = 0$
$(4y-1)(y+3) = 0$
$y = \frac{1}{4}$ or -3
$x^2 = \frac{1}{4}$ or -3
$x = \pm\frac{1}{2}$

6. (i) $x^2 + 4x + 1 = (x+2)^2 - 4 + 1$
$= (x+2)^2 - 3$

(ii) Line of symmetry is x = -2
Vertex (minimum point) is (-2, -3)

(iii)


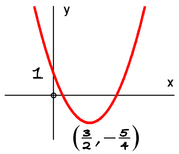
7. (i) $x^2 - 3x + 1 = (x - \frac{3}{2})^2 - \frac{9}{4} + 1$

$= (x - \frac{3}{2})^2 - \frac{5}{4}$

(ii) Lir $= \frac{3}{2}$

Ver) is $(\frac{3}{2}, -\frac{5}{4})$

(iii)



Section 1: Quadratic graphs and equations

Solutions to Exercise level 3

1. (i) $A = x(x+4) + \frac{1}{2}x(x+2)$

$= x^2 + 4x + \frac{1}{2}x^2 + x$

$= \frac{3}{2}x^2 + 5x$

(ii) $P = 2(x+4) + x + (x+2) + \sqrt{x^2 + (x+2)^2}$

$= 4x + 10 + \sqrt{x^2 + (x+2)^2}$

(iii) $A = \frac{3}{2}x^2 + 5x = 200$

$\Rightarrow 3x^2 + 10x - 400 = 0$

$\Rightarrow (x-10)(3x+40) = 0$

$\Rightarrow x = 10$

$\Rightarrow P = 4x + 10 + \sqrt{x^2 + (x+2)^2}$

$= 40 + 10 + \sqrt{100 + 144}$

$= 65.6$ m (3 s.f.)

2. If x is the number of seats in the original rows, then $x = \frac{300}{n}$

and in the extension $\left(2 + \frac{n}{2}\right)(x - 5) = 120$

$\Rightarrow (4+n)\left(\frac{300}{n} - 5\right) = 240$

$\Rightarrow (4+n)(300 - 5n) = 240n$

$\Rightarrow 1200 + 300n - 20n - 5n^2 = 240n$

$\Rightarrow 5n^2 - 40n - 1200 = 0$

$\Rightarrow (5n + 60)(n - 20) = 0$

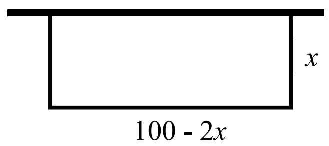
$\Rightarrow n = 20$

So there are 20 rows of 15 seats in the first block, and 12 rows of 10 seats in the new extension.

3. (i) If the width of the enclosure is x, then the length is given by

length = 100 - 2x

The area is given by



$y = x(100 - 2x)$

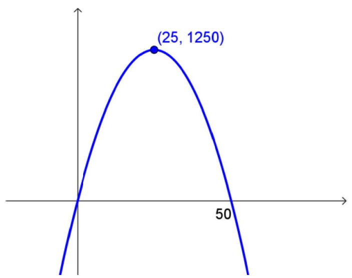
$= -2(x^2 - 50x)$

(ii) So $y = -2(x^2 - 50x)$

$= -2\left[(x-25)^2 - 625\right]$

and for the maximum area, $x = 25$ m, so the enclosure measures 25 metres wide by 50 metres long, and the area is 1250 m².

(iii)



At the intercepts on the x-axis, either $x = 0$ or $x = 50$, and in either case, the 'rectangle' has a zero area.

Section 2: The quadratic formula

Solutions to Exercise level 1

1. (i) $3x^2 + 2x + 5 = 0 \Rightarrow$ discriminant $= b^2 - 4ac$

$= 4 - 4.3.5 < 0$

so no solutions

(ii) $2x^2 - 3x - 2 = 0 \Rightarrow$ discriminant $= b^2 - 4ac$

$= 9 - 4(2)(-2) > 0$

so two solutions

(iii) $5x^2 - 6 = 0 \Rightarrow$ discriminant $= b^2 - 4ac$

$= 0 - 4.5(-6) > 0$

so two solutions

(iv) $4x^2 - 8x + 4 = 0 \Rightarrow$ discriminant $= b^2 - 4ac$

$= 64 - 4(4)(4) = 0$

so two (equal) solutions

(v) $x^2 - 3x + 3 = 0 \Rightarrow$ discriminant $= b^2 - 4ac$

$= 9 - 4.1.3 < 0$

so no solutions

(vi) $-5x^2 - 8x - 10 \Rightarrow$ discriminant $= b^2 - 4ac$

$= 64 - 4(-5)(-10) < 0$

so no solutions

2. (i) $a = 1, b = 4, c = 1$

$$x = \frac{-4 \pm \sqrt{4^2 - 4 \times 1 \times 1}}{2 \times 1}$$

$$= \frac{-4 \pm \sqrt{12}}{2}$$

$$= \frac{-4 \pm 2\sqrt{3}}{2}$$

$$= -2 \pm \sqrt{3}$$

(ii) $a=1, b=-3, c=-1$

$$x=\frac{3\pm\sqrt{3^2-4\times1\times-1}}{2\times1}$$
$$=\frac{3\pm\sqrt{13}}{2}$$

(iii) $a=2, b=2, c=-3$

$$x=\frac{-2\pm\sqrt{2^2-4\times2\times-3}}{2\times2}$$
$$=\frac{-2\pm\sqrt{28}}{4}$$
$$=\frac{-2\pm2\sqrt{7}}{4}$$
$$=\frac{-1\pm\sqrt{7}}{2}$$

(iv) $a=3, b=-4, c=-2$

$$x=\frac{4\pm\sqrt{4^2-4\times3\times-2}}{2\times3}$$
$$=\frac{4\pm\sqrt{40}}{6}$$
$$=\frac{4\pm2\sqrt{10}}{6}$$
$$=\frac{2\pm\sqrt{10}}{3}$$

Section 2: The quadratic formula

Solutions to Exercise level 2

1. (i) $x^2+2x-2=0$

$a=1, b=2, c=-2$

Discriminant $=b^2-4ac=2^2-4\times1\times-2=4+8=12$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-2\pm\sqrt{12}}{2}=\frac{-2\pm2\sqrt{3}}{2}=-1\pm\sqrt{3}$$

(ii) $x^2-3x+5=0$

$a=1, b=-3, c=5$

Discriminant $=b^2-4ac=(-3)^2-4\times1\times5=9-20=-11$

The discriminant is negative so there are no real roots.

(iii) $2x^2+x-4=0$

$a=2, b=1, c=-4$

Discriminant $=b^2-4ac=1^2-4\times2\times-4=1+32=33$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-1\pm\sqrt{33}}{2\times2}=\frac{-1\pm\sqrt{33}}{4}$$

(iv) $2x^2-5x-12=0$

$a=2, b=-5, c=-12$

Discriminant $=b^2-4ac=(-5)^2-4\times2\times-12=25+96=121$

Since the discriminant is a perfect square, then it is possible to factorise.

$(2x+3)(x-4)=0$

$x=-\frac{3}{2}$ or $x=4$

(v) $x^2-5x-3=0$

$a=1, b=-5, c=-3$

Discriminant $=b^2-4ac=(-5)^2-4\times1\times-3=25+12=37$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{5\pm\sqrt{37}}{2}$$

(vi) $3x^2+x+1=0$

$a=3, b=1, c=1$

Discriminant $=b^2-4ac=1^2-4\times3\times1=1-12=-11$

The discriminant is negative so there are no real roots.

(vii) $4x^2+12x+9=0$

$a=4, b=12, c=9$

Discriminant $= b^2 - 4ac = 12^2 - 4\times 4\times 9 = 144 - 144 = 0$
Since the discriminant is zero, there is a repeated root and the equation can be factorised.
$(2x+3)^2 = 0$
$x = -\frac{3}{2}$

(viii) $4x^2 + 10x + 5 = 0$
$a = 4, b = 10, c = 5$
Discriminant $= b^2 - 4ac = 10^2 - 4\times 4\times 5 = 100 - 80 = 20$
$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-10 \pm \sqrt{20}}{2\times 4} = \dfrac{-10 \pm 2\sqrt{5}}{8} = \dfrac{-5 \pm \sqrt{5}}{4}$

2. (i) $x = \dfrac{3}{x} - 1$
Multiplying through by x: $x^2 = 3 - x$
$x^2 + x - 3 = 0$
$x = \dfrac{-1 \pm \sqrt{1^2 - 4\times 1\times -3}}{2\times 1}$
$= \dfrac{-1 \pm \sqrt{13}}{2}$

(ii) $6\sqrt{x} - 7 = x$
Substituting $\sqrt{x} = y$ gives $6y - 7 = y^2$
$y^2 - 6y + 7 = 0$
$y = \dfrac{6 \pm \sqrt{6^2 - 4\times 1\times 7}}{2\times 1}$
$= \dfrac{6 \pm \sqrt{8}}{2}$
$= \dfrac{6 \pm 2\sqrt{2}}{2}$
$= 3 \pm \sqrt{2}$

$x = y^2 = \left(3 \pm \sqrt{2}\right)^2$
$= 9 \pm 6\sqrt{2} + 2$
$= 11 \pm 6\sqrt{2}$

3. Surface area of cylinder $= 2\pi r^2 + 2\pi rh$
$= 2\pi r^2 + 40\pi r$
$2\pi r^2 + 40\pi r = 300$
$\pi r^2 + 20\pi r - 150 = 0$
$r = \dfrac{-20\pi \pm \sqrt{(20\pi)^2 - 4\times \pi \times -150}}{2\times \pi}$
$= 2.16$ or -22.2
Since the radius must be positive, $r = 2.16$.

4. If there is a repeated root, the discriminant is zero.
$a = 1, b = (3k+1), c = (4k+13)$
$b^2 - 4ac = 0$
$(3k+1)^2 - 4(4k+13) = 0$
$9k^2 + 6k + 1 - 16k - 52 = 0$
$9k^2 - 10k - 51 = 0$
$(k-3)(9k+17) = 0$
$k = 3$ or $k = -\frac{17}{9}$

5. If there is at least one real root, the discriminant is greater than or equal to zero.
$a = 2, b = -5, c = k$
$b^2 - 4ac \geq 0$
$(-5)^2 - 4\times 2\times k \geq 0$
$25 - 8k \geq 0$
$k \leq \frac{25}{8}$
The greatest possible value of k is $\frac{25}{8}$.

Section 2: The quadratic formula

Solutions to Exercise level 3

1. The stone passes the clifftop when $h = 0$,

$$h = 0 \Rightarrow 0 = 20t - 5t^2$$
$$\Rightarrow 5t(t-4) = 0$$
$$\Rightarrow t = 0,\ t = 4$$

So the stone passes the clifftop on the way downwards after 4 seconds.

The stone reaches the sea when $h = -50$,

$$h = -50 \Rightarrow 5t^2 - 20t - 50 = 0$$
$$\Rightarrow \quad t^2 - 4t - 10 = 0$$
$$\Rightarrow \quad t = \frac{4 \pm \sqrt{16+40}}{2}$$
$$\Rightarrow \quad t = 5.74,\ -1.74$$

The stone hits the sea after 5.74 seconds (to 3 sig. figs.).

The negative root can be interpreted as the time before the stone was thrown when it should have been thrown from sea level to follow the same path.

2. (i) $x^2 + 8x + c = (x+4)^2 + (c-16)$

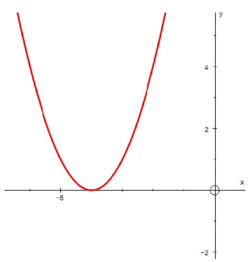
(ii) If the equation has real roots, then the vertex of the graph must be on or below the x – axis, so $c - 16 \le 0$

$$\Rightarrow \quad c \le 16$$

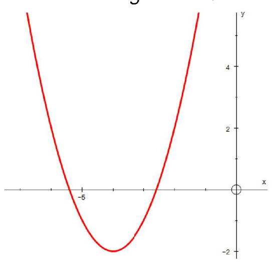
(iii) If the equation has unequal real roots, then the vertex must be strictly below the x – axis, so $c - 16 < 0$

$$\Rightarrow \quad c < 16$$

(iv) Two real equal roots
$c = 16$



Two real unequal roots
e.g. $c = 14$



(v) In the two graphs, the vertex in each case lies on the line $x = -4$ and this will be the case for all values of c. So there will never be a graph with two intercepts with the x-axis which are both positive.

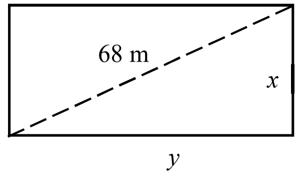
3. (i) In the diagram

$$2x + 2y = 184$$
$$\Rightarrow \quad y = 92 - x$$

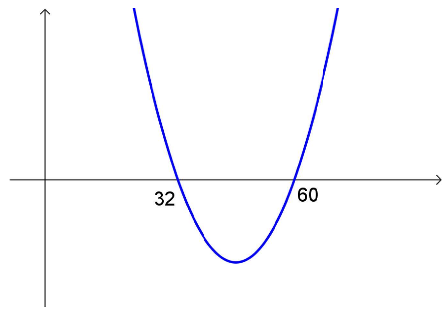
By Pythagoras' Theorem,

$$x^2 + y^2 = 68^2$$
$$\Rightarrow \quad x^2 + (92-x)^2 = 68^2$$
$$\Rightarrow 2x^2 - 184x + 3840 = 0$$
$$\Rightarrow \quad x^2 - 92x + 1920 = 0$$
$$\Rightarrow \quad (x-32)(x-60) = 0$$

So the car park measures 32 metres by 60 metres.



(ii)



The intersections give both dimensions of the car park.

Section 1: Simultaneous equations

Solutions to Exercise level 1

1. (i) $2x + 5y = 11$ (1)

$2x - y = 5$ (2)

Subtracting: $6y = 6$

$y = 1$

Substituting into (1): $2x + 5 \times 1 = 11$

$2x = 6$

$x = 3$

The solution is $x = 3, y = 1$. Check: $2x + 5y = 2 \times 3 + 5 \times 1 = 11$

$2x - y = 2 \times 3 - 1 = 5$

(ii) $x + 2y = 6$ (1)×4 $4x + 8y = 24$

$4x + 3y = 4$ (2) $4x + 3y = 4$

Subtracting: $5y = 20$

$y = 4$

Substituting into (1): $x + 2 \times 4 = 6$

$x = -2$

The solution is $x = -2, y = 4$. Check: $x + 2y = -2 + 8 = 6$

$4x + 3y = -8 + 12 = 4$

(iii) $3a - 2b = 4$ (1)×2 $6a - 4b = 8$

$5a + 4b = 3$ (2) $5a + 4b = 3$

Adding: $11a = 11$

$a = 1$

Substituting into (1): $3 \times 1 - 2b = 4$

$-2b = 1$

$b = -\frac{1}{2}$

The solution is $a = 1, b = -\frac{1}{2}$. Check: $3a - 2b = 3 + 1 = 4$

$5a + 4b = 5 - 2 = 3$

(iv) $2p - 5q = 5$ (1)×3 $6p - 15q = 15$

$3p - 2q = -9$ (2)×2 $6p - 4q = -18$

Subtracting: $-11q = 33$

$q = -3$

Substituting into (1): $2p - 5 \times -3 = 5$

$2p = -10$

$p = -5$

The solution is $p = -5, q = -3$. Check: $2p - 5q = -10 + 15 = 5$

$3p - 2q = -15 + 6 = -9$

(v) $5x + 3y = 9$ (1)

$y = 3x - 4$ (2)

Substituting (2) into (1): $5x + 3(3x - 4) = 9$

$5x + 9x - 12 = 9$

$14x = 21$

$x = \frac{3}{2}$

Substituting into (1): $y = 3 \times \frac{3}{2} - 4 = \frac{9}{2} - 4 = \frac{1}{2}$

The solution is $x = \frac{3}{2}, y = \frac{1}{2}$. Check: $5x + 3y = \frac{15}{2} + \frac{3}{2} = 9$

(vi) $3a + 2b = 1$ (1)×2 $6a + 4b = 2$

$9a - 4b = 4$ (2) $9a - 4b = 4$

Adding: $15a = 6$

$a = \frac{2}{5}$

Substituting into (1): $3 \times \frac{2}{5} + 2b = 1$

$2b = 1 - \frac{6}{5} = -\frac{1}{5}$

$b = -\frac{1}{10}$

The solution is $a = \frac{2}{5}, b = -\frac{1}{10}$. Check: $3a + 2b = \frac{6}{5} - \frac{1}{5} = 1$

$9a - 4b = \frac{18}{5} + \frac{2}{5} = 4$

2. (i) $x - y = -1$ (A)

$3x + 2y = 7$ (B)

(A) $\Rightarrow 2x - 2y = -2$ (C)

(C) + (B) $\Rightarrow 5x = 5$

$\Rightarrow x = 1, y = 2$

(ii) $2x + y = 0$ (A)

$x - 3y = 7$ (B)

(B) $\Rightarrow 2x - 6y = 14$ (C)

(A) - (C) $\Rightarrow 7y = -14$

$\Rightarrow y = -2, x = 1$

(iii) $y - 5x = -8$ (A)

$x + 3y = 0$ (B)

(B) $\Rightarrow 5x + 15y = 0$ (C)

(A) + (C) $\Rightarrow 16y = -8$

$\Rightarrow y = -\frac{1}{2},\ x = \frac{3}{2}$

(iv) $x = 2y - 1$ (A)

$-x + 3y = -1$ (B)

(A) + (B) $\Rightarrow 3y = 2y - 2$ (C)

$\Rightarrow y = -2,\ x = -5$

(v) $2p - 4q = 14$ (A)

$-p + 3q = -5$ (B)

(B) $\Rightarrow -2p + 6q = -10$ (C)

(A) + (C) $\Rightarrow 2q = 4$

$\Rightarrow q = 2,\ p = 11$

(vi) $3u - 2v = -26$ (A)

$-u + 6v = 46$ (B)

(B) $\Rightarrow -3u + 18v = 138$ (C)

(A) + (C) $\Rightarrow 16v = 112$

$\Rightarrow v = 7,\ u = -4$

Section 1: Simultaneous equations

Solutions to Exercise level 2

1. (i) $7x^2 + y^2 = 64$ (1)

$x + y = 4$ (2)

(2) $\Rightarrow y = 4 - x$

Substituting into (1): $7x^2 + (4 - x)^2 = 64$

$7x^2 + 16 - 8x + x^2 = 64$

$8x^2 - 8x - 48 = 0$

$x^2 - x - 6 = 0$

$(x - 3)(x + 2) = 0$

$x = 3$ or $x = -2$

When $x = 3$, $y = 4 - 3 = 1$

When $x = -2$, $y = 4 - (-2) = 6$

The solutions are $x = 3, y = 1$ and $x = -2, y = 6$

Check: $x = 3, y = 1 \Rightarrow 7x^2 + y^2 = 63 + 1 = 64$

$x = -2, y = 6 \Rightarrow 7x^2 + y^2 = 28 + 36 = 64$

(ii) $3x^2 - 2y^2 = -5$ (1)

$y - x = 1$ (2)

(2) $\Rightarrow y = 1 + x$

Substituting into (1): $3x^2 - 2(1 + x)^2 = -5$

$3x^2 - 2(1 + 2x + x^2) = -5$

$3x^2 - 2 - 4x - 2x^2 = -5$

$x^2 - 4x + 3 = 0$

$(x - 1)(x - 3) = 0$

$x = 1$ or $x = 3$

When $x = 1$, $y = 1 + 1 = 2$

When $x = 3$, $y = 1 + 3 = 4$

The solutions are $x = 1, y = 2$ and $x = 3, y = 4$

Check: $x = 1, y = 2 \Rightarrow 3x^2 - 2y^2 = 3 - 5 = -5$

$x = 3, y = 4 \Rightarrow 3x^2 - 2y^2 = 27 - 32 = -5$

(iii) $p^2 + pq = 2$ (1)

$q - p = 3$ (2)

(2) $\Rightarrow q = 3 + p$

Substituting into (1): $p^2 + p(3+p) = 2$

$p^2 + 3p + p^2 = 2$

$2p^2 + 3p - 2 = 0$

$(2p-1)(p+2) = 0$

$p = \frac{1}{2}$ or $p = -2$

When $p = \frac{1}{2}, q = 3 + \frac{1}{2} = \frac{7}{2}$

When $p = -2, q = 3 - 2 = 1$

The solutions are $p = \frac{1}{2}, q = \frac{7}{2}$ and $p = -2, q = 1$.

Check: $p = \frac{1}{2}, q = \frac{7}{2} \Rightarrow p^2 + pq = \frac{1}{4} + \frac{7}{4} = 2$

$p = -2, q = 1 \Rightarrow p^2 + pq = 4 - 2 = 2$

(iv) $8a^2 - b^2 = 2$ (1)

$2a + b = 1$ (2)

(2) $\Rightarrow b = 1 - 2a$

Substituting into (1): $8a^2 - (1-2a)^2 = 2$

$8a^2 - (1 - 4a + 4a^2) = 2$

$8a^2 - 1 + 4a - 4a^2 = 2$

$4a^2 + 4a - 3 = 0$

$(2a+3)(2a-1) = 0$

$a = -\frac{3}{2}$ or $a = \frac{1}{2}$

When $a = -\frac{3}{2}, b = 1 - 2 \times -\frac{3}{2} = 1 + 3 = 4$

When $a = \frac{1}{2}, b = 1 - 2 \times \frac{1}{2} = 1 - 1 = 0$

The solutions are $a = -\frac{3}{2}, b = 4$ and $a = \frac{1}{2}, b = 0$.

Check: $a = -\frac{3}{2}, b = 4 \Rightarrow 8a^2 - b^2 = 8 \times \frac{9}{4} - 16 = 18 - 16 = 2$\

$a = \frac{1}{2}, b = 0 \Rightarrow 8a^2 - b^2 = 8 \times \frac{1}{4} - 0 = 2$

2. (i) $y = 9 - x \Rightarrow x^2 - 3x(9-x) + 2(9-x)^2 = 0$

$\Rightarrow 2x^2 - 21x + 54 = 0$

$\Rightarrow (2x-9)(x-6) = 0$

$\Rightarrow x = \frac{9}{2}, y = \frac{9}{2}$ or $x = 6, y = 3$

(ii) $y = \frac{8}{x} \Rightarrow 3x - \frac{8}{x} = 10$

$\Rightarrow 3x^2 - 8 = 10x$

$\Rightarrow 3x^2 - 10x - 8 = 0$

$\Rightarrow (x-4)(3x+2) = 0$

$\Rightarrow x = 4, y = 2$ or $x = -\frac{2}{3}, y = -12$

(iii) $y = 4x$

$3y^2 - 2xy = 160$

$\Rightarrow 3(4x)^2 - 2x(4x) = 160$

$\Rightarrow 40x^2 = 160$

$\Rightarrow x = 2, y = 8$ or $x = -2, y = -8$

3. (i) $y = 3x - 2$

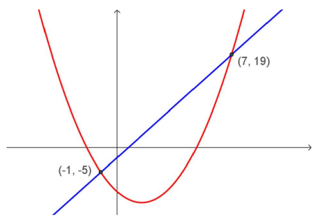
$y = x^2 - 3x - 9$

$\Rightarrow 3x - 2 = x^2 - 3x - 9$

$\Rightarrow x^2 - 6x - 7 = 0$

$\Rightarrow (x-7)(x+1) = 0$

$\Rightarrow x = 7, y = 19$ or $x = -1, y = -5$



(ii) $y + 2x = 3$

$y = 6 + 4x - x^2$

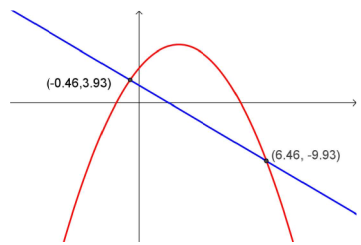
$\Rightarrow -2x + 3 = 6 + 4x - x^2$

$\Rightarrow x^2 - 6x - 3 = 0$

$\Rightarrow x = \frac{6 \pm \sqrt{36+12}}{2}$

$\Rightarrow x = 6.46, y = -9.93$

or $x = -0.46, y = 3.93$



4. (i) $x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$

so vertex is $\left(-\frac{1}{2}, \frac{3}{4}\right)$

(ii) $\left.\begin{array}{l} y = x^2 + x + 1 \\ y = 5x - 3 \end{array}\right\}$ cross when

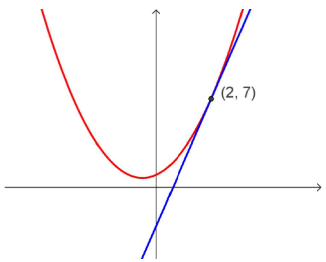
$x^2 + x + 1 = 5x - 3$

$\Rightarrow x^2 - 4x + 4 = 0$

$\Rightarrow \quad (x-2)^2 = 0$

$\Rightarrow \quad x = 2$ (twice!)

(iii) When $x = 2$, then $y = 7$



From the graph, there is exactly one 'crossing' point, at (2 , 7) so the straight line is a tangent to the quadratic graph.

5. $x^2 + kx + 6 = 2x - 3$

$x^2 + (k-2)x + 9 = 0$

$a = 1, b = k-2, c = 9$

If the graphs touch, discriminant $= 0$

$(k-2)^2 - 4 \times 1 \times 9 = 0$

$(k-2)^2 = 36$

$k - 2 = \pm 6$

$k = 2 \pm 6$

$k = -4$ or 8

Section 1: Simultaneous equations

Solutions to Exercise level 3

1. $\left.\begin{array}{l} 4x + 2(\frac{1}{2}x) + 2y = 100 \\ x^2 - (\frac{1}{2}x)y = 279 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 5x + 2y = 100 \\ x^2 - \frac{1}{2}xy = 279 \end{array}\right\}$

$\Rightarrow x^2 - \frac{1}{2}x(50 - \frac{5}{2}x) = 279$

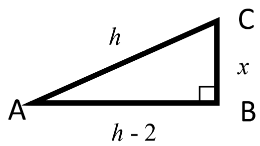
$\Rightarrow 9x^2 - 100x - 1116 = 0$

$\Rightarrow (x-18)(9x+62) = 0$

$\Rightarrow \quad x = 18, y = 5$

So the lawn is 18 m square and the flowerbed is 9 m by 5 m.

2. $\left.\begin{array}{l} \frac{1}{2}x(h-2) = 60 \\ x + (h-2) + h = 40 \end{array}\right\} \Rightarrow \left.\begin{array}{l} \frac{1}{2}x(h-2) = 60 \\ x + 2h = 42 \end{array}\right\}$



$\Rightarrow x = 42 - 2h = 2(21-h)$

$\Rightarrow \frac{1}{2} \times 2(21-h)(h-2) = 60$

$\Rightarrow -h^2 + 23h - 42 = 60$

$\Rightarrow h^2 - 23h + 102 = 0$

$\Rightarrow (h-17)(h-6) = 0$

$\Rightarrow h = 17, x = 8$ or $h = 6, x = 30$

h cannot be less than x, so h = 17 and x = 8

So the triangle has AB length 15 m, BC length 8 m and AC 17 m

3. $A = 2(30x) + 2(30y) + xy$

$= 60x + 60y + xy$

$V = 30xy$

$V = 45000 \Rightarrow y = \dfrac{1500}{x}$

$A = 6300 \Rightarrow 60x + 60\left(\frac{1500}{x}\right) + x\left(\frac{1500}{x}\right) = 6300$

$\Rightarrow x^2 - 80x + 1500 = 0$

$\Rightarrow (x-50)(x-30) = 0$

$\Rightarrow x = 50, y = 30$ or $x = 30, y = 50$

So the dimensions of the tank are 50 cm x 30 cm x 30 cm.

Section 2: Inequalities

Solutions to Exercise level 1

1. (i) $2x+3<10$
$2x<7$
$x<\frac{7}{2}$

(ii) $5x+3\geq 2x-9$
$3x+3\geq -9$
$3x\geq -12$
$x\geq -4$

(iii) $3x-1>7-x$
$4x-1>7$
$4x>8$
$x>2$

(iv) $4x+1\leq 6x-7$
$1\leq 2x-7$
$8\leq 2x$
$4\leq x$
$x\geq 4$

(v) $5x+2>-7$
$5x>-9$
$x>-\frac{9}{5}$

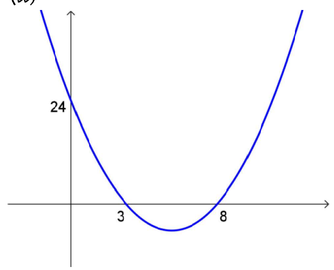
(vi) $3x-11\leq 5+4x$
$-16\leq x$
$x\geq -16$

(vii) $3(2-3x)\geq 5x+1$
$6-9x\geq 5x+1$
$5\geq 14x$
$x\leq \frac{5}{14}$

(viii) $\frac{1}{3}(7+6x)<2-x$
$7+6x<6-3x$
$9x<-1$
$x<-\frac{1}{9}$

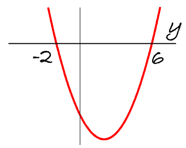
2. (i) $x^2-11x+24=(x-8)(x-3)$

(ii)

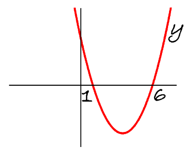


(iii) From the graph, $x^2-11x+24\geq 0 \Rightarrow x\leq 3$ or $x\geq 8$.

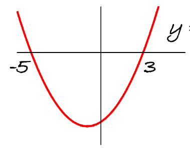
3. (i) $x^2-4x-12\leq 0$
$(x-6)(x+2)\leq 0$
From graph, $-2\leq x\leq 6$



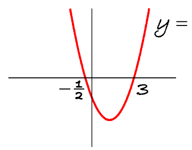
(ii) $x^2-7x+6>0$
$(x-1)(x-6)>0$
From graph, $x<1$ or $x>6$



(iii) $x^2+2x-15\geq 0$
$(x+5)(x-3)\geq 0$
From graph, $x\leq -5$ or $x\geq 3$



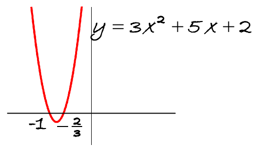
(iv) $2x^2-5x-3\leq 0$
$(2x+1)(x-3)\leq 0$
From graph, $-\frac{1}{2}\leq x\leq 3$



(v) $3x^2 + 5x + 2 < 0$

$(3x+2)(x+1) < 0$

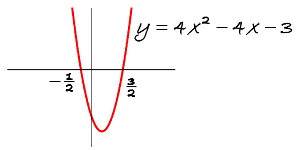
From graph, $-1 < x < -\frac{2}{3}$



(vi) $4x^2 - 4x - 3 > 0$

$(2x-3)(2x+1) > 0$

From graph, $x < -\frac{1}{2}$ or $x > \frac{3}{2}$



4. (i) $x > 2$

Since the line is dotted, $x = 2$ is not included in the region

(ii) $y \geq 1$

(iii) $y > x$

(iv) $y \leq -x + 1$

Section 2: Inequalities

Solutions to Exercise level 2

1. (i) $5(x-3) \leq 2(2x+3)$

$5x - 15 \leq 4x + 6$

$x - 15 \leq 6$

$x \leq 21$

(ii) $2(1-x) > 3x + 4$

$2 - 2x > 3x + 4$

$2 > 5x + 4$

$-2 > 5x$

$-\frac{2}{5} > x$

$x < -\frac{2}{5}$

(iii) $4(2x+5) \geq 3(3x-1)$

$8x + 20 \geq 9x - 3$

$20 \geq x - 3$

$23 \geq x$

$x \leq 23$

(iv) $\frac{2x+1}{3} > \frac{x-4}{2}$

$2(2x+1) > 3(x-4)$

$4x + 2 > 3x - 12$

$x + 2 > -12$

$x > -14$

(v) $-\frac{1}{2}(4+3x) \geq 2x - 1$

$4 + 3x \leq -4x + 2$

$7x \leq -2$

$x \leq -\frac{2}{7}$

(vi) $\frac{x-1}{3} > -\frac{3-x}{2}$

$2x - 2 > -9 + 3x$

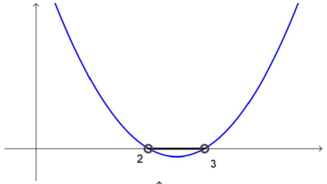
$-x > -7$

$x < 7$

2. (i) $x^2-5x+6=0$

$(x-2)(x-3)=0$

From graph, solution is $2<x<3$

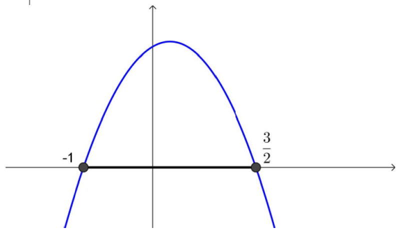


(ii) $-2x^2+x+3=0$

$2x^2-x-3=0$

$(2x-3)(x+1)=0$

From graph, solution is $-1\le x\le \frac{3}{2}$



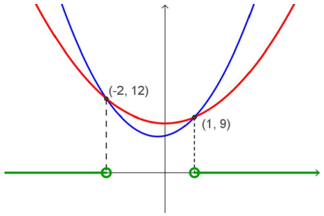
(iii) $x^2+8=2x^2+x+6$

$x^2+x-2=0$

$(x-1)(x+2)=0$

So intersections are (1, 9) and (-2, 12),

From graph, solution is $x<-2$ or $x>1$

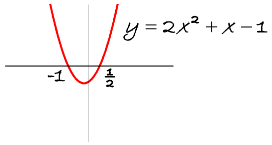


3. (i) $1-x-2x^2\ge 0$

$2x^2+x-1\le 0$

$(2x-1)(x+1)\le 0$

From graph, $-1\le x\le \frac{1}{2}$



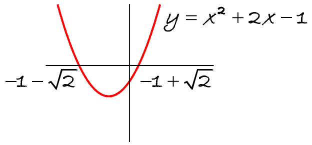
(ii) $x^2+2x-1<0$ cannot be factorised, so use quadratic formula to solve the equation $x^2+2x-1=0$:

$a=1, b=2, c=-1$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-2\pm\sqrt{2^2-4\times1\times-1}}{2}$$

$$=\frac{-2\pm\sqrt{8}}{2}=\frac{-2\pm2\sqrt{2}}{2}=-1\pm\sqrt{2}$$

From graph, $-1-\sqrt{2}<x<-1+\sqrt{2}$

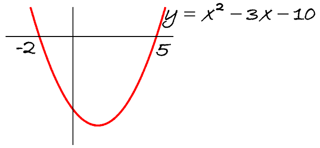


(iii) $x^2\ge 3x+10$

$x^2-3x-10\ge 0$

$(x-5)(x+2)\ge 0$

From graph, $x\le -2$ or $x\ge 5$



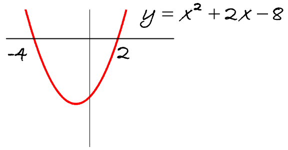
(iv) $x(x+3)>x+8$

$x^2+3x>x+8$

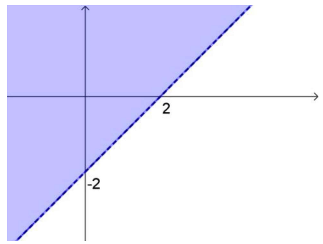
$x^2+2x-8>0$

$(x+4)(x-2)>0$

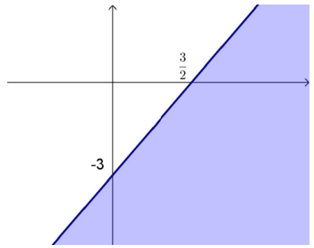
From graph, $x<-4$ or $x>2$



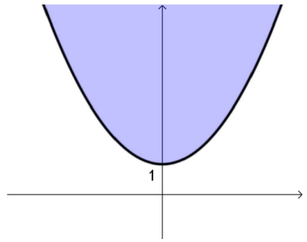
4. (i)



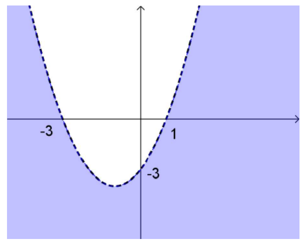
(ii)



(iii)



(iv)



5. There are no real roots if the discriminant is less than zero.

(i) $x^2-5x+k=0$

$a=1, b=-5, c=k$

$b^2-4ac<0$

$(-5)^2-4\times1\times k<0$

$25-4k<0$

$25<4k$

$4k>25$

$k>\frac{25}{4}$

(ii) $x^2 + kx + k + 3 = 0$

$a = 1, b = k, c = k + 3$

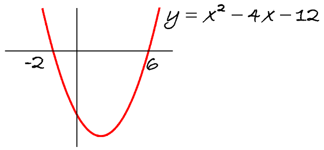
$b^2 - 4ac < 0$

$k^2 - 4 \times 1 \times (k + 3) < 0$

$k^2 - 4k - 12 < 0$

$(k - 6)(k + 2) < 0$

From graph, $-2 < k < 6$.



Section 2: Inequalities

Solutions to Exercise level 3 (Extension)

1. If John's age is x years, and his mother's age is y years, then

$x < \frac{1}{2}y$ (1)

$x + y > 60$ (2)

$y = 26 + x$ (3)

Substituting (3) into (1) $\Rightarrow 2x < 26 + x$

$\Rightarrow x < 26$

Substituting (3) into (2) $\Rightarrow x + (26 + x) > 60$

$\Rightarrow 2x > 34$

$\Rightarrow x > 17$

so John's age is between 18 and 25 inclusive.

2. Area: $x(x-3) \le 88$

Perimeter: $2x + 2(x-3) \ge 30$

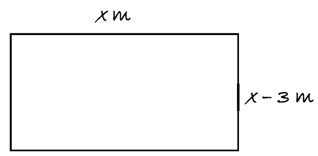
$\Rightarrow \left.\begin{array}{ll} x^2 - 3x - 88 \le 0 & (1) \\ 4x - 36 \ge 0 & (2) \end{array}\right\}$

(1) $\Rightarrow (x-11)(x+8) \le 0$

$\Rightarrow x \le 11$ (precisely: $-8 \le x \le 11$)

(2) $\Rightarrow x \ge 9$

so the length of the room is between 9 and 11 metres.



3. Volume of cone $= \frac{1}{3}\pi r^2 h$

$\Rightarrow \frac{1}{3}\pi r^2 h \le 25$ (1)

For the slant height, $l < 2r$

and $l^2 = r^2 + h^2$

$\Rightarrow r^2 + h^2 < 4r^2$

$\Rightarrow r^2 > \frac{1}{3}h^2$

So (1) $\Rightarrow \frac{1}{9}\pi h^3 < 25$

$\Rightarrow h^3 < \frac{225}{\pi}$

$\Rightarrow h < 4.15$ m (3 s.f.)