Section 2: Matrix transformations

Exercise

1. The unit square OABC has vertices O (0, 0), A (1, 0), B (1, 1) and C (0, 1).
For each of the following matrices, find the image of each point, and describe the transformation.

   (i) $\begin{pmatrix} -0.5 & 0 \\ 0 & -0.5 \end{pmatrix}$

   (ii) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

2. Find the images of A (3, 1), B (3, 3), C(6, 3), D(6, 1) under the transformation $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$.

3. Find the images of P (0, 0), Q (1, 1), R (0, 2) and S (-1, 1) under the transformation $\begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}$.

4. Find 2×2 matrices to represent the transformations **P**, which is a reflection in the y axis and **Q**, which is a rotation of 90° clockwise about the origin. Hence find a single matrix to represent a reflection in the y axis followed by a rotation of 90° clockwise about the origin. Describe this as a single transformation.

5. The unit square with vertices (0, 0), (1, 0), (1, 1) and (0, 1) is transformed to give a square of area 2.

   Find two possible matrices which represent transformations which will do this.

6. The transformation represented by the matrix $\begin{pmatrix} 5 & 2 \\ 4 & a \end{pmatrix}$ transforms the point (1, -2) to $(k, \; -2k)$.
Find the values of a and k.

Section 2: Matrix transformations

Solutions to Exercise

1. (i) Image of O = (0, 0)
Image of A = (-0.5, 0)
Image of B = (-0.5, -0.5)
Image of C = (0, -0.5)

The transformation is an enlargement, scale factor -0.5, centre the origin.

(ii) Image of O = (0, 0)
Image of A = (0, -1)
Image of B = (-1, -1)
Image of C = (-1, 0)

The transformation is a reflection in the line y = -x.

2. A: $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ so the image of A is (3, -5)

B: $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$ so the image of B is (3, -3)

C: $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -9 \end{pmatrix}$ so the image of C is (6, -9)

D: $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -11 \end{pmatrix}$ so the image of D is (6, -11)

3. P: $\begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so the image of P is (0, 0)

Q: $\begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ -5 \end{pmatrix}$ so the image of Q is (7, -5)

R: $\begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \end{pmatrix}$ so the image of R is (6, -4)

S: $\begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ so the image of S is (-1, 1)

AQA FM Matrices 2 Exercise solutions

4. Under P, the point (1, 0) is mapped to the point (-1, 0) and the point (0, 1) is unchanged.

   So P is represented by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

   Under Q, the point (1, 0) is mapped to the point (0, -1) and the point (0, 1) is mapped to the point (1, 0).

   So Q is represented by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

   The single matrix is $QP = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

   This transformation is a reflection in the line $y = x$.

5. Enlargements with either scale factor $\sqrt{2}$ or $\sqrt{2}$ would work.

   The relevant matrices are $\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$ and $\begin{pmatrix} -\sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix}$.

6. $\begin{pmatrix} 5 & 2 \\ 4 & a \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} k \\ -2k \end{pmatrix}$

   $5 - 4 = k$

   $4 - 2a = -2k$

   $k = 1,\ a = 3$

Exercise 2

1 [Jan 2013 Paper 2 Q15] Describe fully the **single** transformation represented by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

?

2 [Set 2 Paper 1 Q4] The transformation matrix $\begin{pmatrix} a & 2 \\ -1 & 1 \end{pmatrix}$ maps the point $(3,4)$ onto the point $(2, b)$. Work out the values of a and b.

?

3 [Set 3 Paper 1 Q6] The matrix $\begin{pmatrix} a & b \\ -a & 2b \end{pmatrix}$ maps the point $(5,4)$ onto the point $(1,17)$. Work out the values of a and b.

?

4 [Worksheet 2 Q5] Work out the image of the point D (–1, 2) after transformation by the matrix $\begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}$

?

5 [Worksheet 2 Q6] The point $A(m, n)$ is transformed to the point A' (–2, 0) by the matrix $\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$

Work out the values of m and n.

?

6 [Worksheet 2 Q8] Describe fully the transformation given by the matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

?

7 [Worksheet 2 Q9] The unit square $OABC$ is transformed by the matrix $\begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}$ to the square $OA'B'C'$.

The area of $OA'B'C'$ is 27. Work out the exact value of h.

?

Exercise 2

1 [Jan 2013 Paper 2 Q15] Describe fully the **single** transformation represented by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Rotation 90° anticlockwise about the origin.

2 [Set 2 Paper 1 Q4] The transformation matrix $\begin{pmatrix} a & 2 \\ -1 & 1 \end{pmatrix}$ maps the point $(3,4)$ onto the point $(2, b)$. Work out the values of a and b.

$$\boldsymbol{a = -2, b = 1}$$

3 [Set 3 Paper 1 Q6] The matrix $\begin{pmatrix} a & b \\ -a & 2b \end{pmatrix}$ maps the point $(5,4)$ onto the point $(1,17)$. Work out the values of a and b.

$$\boldsymbol{a = -1, b = 1.5}$$

4 [Worksheet 2 Q5] Work out the image of the point *D* (–1, 2) after transformation by the matrix $\begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}$ **Solution:** $\begin{pmatrix} \mathbf{4} \\ \mathbf{3} \end{pmatrix}$

5 [Worksheet 2 Q6] The point *A*(*m*, *n*) is transformed to the point *A′* (–2, 0) by the matrix $\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$

Work out the values of *m* and *n*.

$$\boldsymbol{m = 2, n = -2}$$

6 [Worksheet 2 Q8] Describe fully the transformation given by the matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

Reflection in the line $\boldsymbol{y = -x}$

7 [Worksheet 2 Q9] The unit square *OABC* is transformed by the matrix $\begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}$ to the square *OA′B′C′*. The area of *OA′B′C′* is 27. Work out the exact value of *h*.

$$\boldsymbol{h = 3\sqrt{3}}$$

Exercise 3

1 Point $(3,-2)$ is transformed by the matrix $\begin{pmatrix}1 & -1\\0 & 1\end{pmatrix}$ followed by a further transformation by the matrix $\begin{pmatrix}0 & 2\\1 & 0\end{pmatrix}$.

(i) Work out the matrix for the combined transformation.

?

(ii) Work out the co-ordinates of the image point of P.

?

2 Point $(-1,4)$ is transformed by the matrix $\begin{pmatrix}3 & -1\\-2 & 2\end{pmatrix}$ followed by a further transformation by the matrix $\begin{pmatrix}1 & 0\\3 & -2\end{pmatrix}$.

(i) Work out the matrix for the combined transformation.

?

(ii) Work out the co-ordinates of the image point of W.

?

3 The unit square is reflected in the x-axis followed by a rotation through $180°$ centre the origin. Work out the matrix for the combined transformation.

?

4 The unit square is enlarged, centre the origin, scale factor 2 followed by a reflection in the line $y = x$. Work out the matrix for the combined transformation.

?

Exercise 3

1 Point $(3, -2)$ is transformed by the matrix $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ followed by a further transformation by the matrix $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$.

(i) Work out the matrix for the combined transformation. **Solution:** $\begin{pmatrix} \mathbf{0} & \mathbf{2} \\ \mathbf{1} & \mathbf{-1} \end{pmatrix}$

(ii) Work out the co-ordinates of the image point of P. **Solution:** $(\mathbf{-4}, \mathbf{5})$

2 Point $(-1,4)$ is transformed by the matrix $\begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix}$ followed by a further transformation by the matrix $\begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix}$.

(i) Work out the matrix for the combined transformation. **Solution:** $\begin{pmatrix} \mathbf{3} & \mathbf{-1} \\ \mathbf{13} & \mathbf{-7} \end{pmatrix}$

(ii) Work out the co-ordinates of the image point of W. **Solution:** $(\mathbf{-7}, \mathbf{-41})$

3 The unit square is reflected in the x-axis followed by a rotation through $180°$ centre the origin. Work out the matrix for the combined transformation.

Solution: $\begin{pmatrix} \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$

4 The unit square is enlarged, centre the origin, scale factor 2 followed by a reflection in the line $y = x$. Work out the matrix for the combined transformation.

Solution: $\begin{pmatrix} \mathbf{0} & \mathbf{2} \\ \mathbf{2} & \mathbf{0} \end{pmatrix}$

Exercise 3

5 [Jan 2013 Paper 2 Q17] $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ represents a reflection in the y-axis. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ represents a reflection in the line $y = x$.
Work out the matrix that represents a reflection in the y-axis followed by a reflection in the line $y = x$.

?

6 [June 2012 Paper Q22] The transformation matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ maps a point P to Q. The transformation matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ maps point Q to point R.
Point R is $(-4,3)$. Work out the coordinates of point P.

?

7 [Set 1 Paper Q14b] The unit square OABC is transformed by reflection in the line $y = x$ followed by enlargement about the origin with scale factor 2. What is the matrix of the combined transformation?

?

$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.
The point $P(2,7)$ is transformed by matrix BA to P'. Show that P' lies on the line
$7x + 2y = 0$.

?

Exercise 3

5 [Jan 2013 Paper 2 Q17] $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ represents a reflection in the y-axis. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ represents a reflection in the line $y = x$.
Work out the matrix that represents a reflection in the y-axis followed by a reflection in the line $y = x$.

$$\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-1} & \mathbf{0} \end{pmatrix}$$

6 [June 2012 Paper Q22] The transformation matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ maps a point P to Q. The transformation matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ maps point Q to point R.
Point R is $(-4,3)$. Work out the coordinates of point P.

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix}\begin{pmatrix} \mathbf{0} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$$

This is a rotation 90° anticlockwise. So original point P is $\begin{pmatrix} \mathbf{3} \\ \mathbf{4} \end{pmatrix}$

7 [Set 1 Paper Q14b] The unit square OABC is transformed by reflection in the line $y = x$ followed by enlargement about the origin with scale factor 2. What is the matrix of the combined transformation?

$$\begin{pmatrix} \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{2} \end{pmatrix}\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{2} \\ \mathbf{2} & \mathbf{0} \end{pmatrix}$$

$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.
The point $P(2,7)$ is transformed by matrix BA to P'. Show that P' lies on the line
$7x + 2y = 0$.

$$\begin{pmatrix} \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}\begin{pmatrix} \mathbf{3} & \mathbf{0} \\ \mathbf{0} & \mathbf{3} \end{pmatrix}\begin{pmatrix} \mathbf{2} \\ \mathbf{7} \end{pmatrix} = \begin{pmatrix} \mathbf{-6} \\ \mathbf{21} \end{pmatrix}$$

$$\mathbf{7(-6) + 2(21) = 0}$$

Review Questions

14 Here are two transformations.

A Rotation 90° clockwise about the origin.

B Reflection in the line $y = x$

Use matrix multiplication to work out the single matrix which represents the combined transformation A followed by B.

[4 marks]

10 The transformation matrix $\begin{pmatrix} a & b \\ 2a & 3b \end{pmatrix}$ maps the point (1, –3) onto the point (1, 4)

Work out the values of a and b.

You **must** show your working.

[5 marks]

Solutions

14	(A =) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	B1	
	(B =) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	B1	
	(BA =) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	M1	must be two 2x2 matrices in the correct order
	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	A1	only if M1 awarded for correct product

Additional Guidance

Mark positively for the B marks (you may see more than 2 matrices)	
If both matrices wrong but then in the correct order	B0, B0, M1, A0
Both matrices correct but in wrong order	B1, B1, M0, A0
Possible to score B1 B0 M1 A0 if one correct and one not	B1, B0, M1, A0
Either A or B on answer line but not identified and no other working	B0, B0, M0, A0
Condone matrices written without brackets throughout	

10	$\begin{pmatrix} a & b \\ 2a & 3b \end{pmatrix}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$	M1	oe implied by a correct equation
	$a - 3b = 1$	A1	oe may be implied by correct answers
	$2a - 9b = 4$	A1	
	Correct elimination of a variable from their 2 linear equations with both equations having the same two variables	M1	eg $3a - 2a = 3 - 4$ or $-6b - -9b = 2 - 4$
	$a = -1$ $b = -\frac{2}{3}$	A1	must be exact values