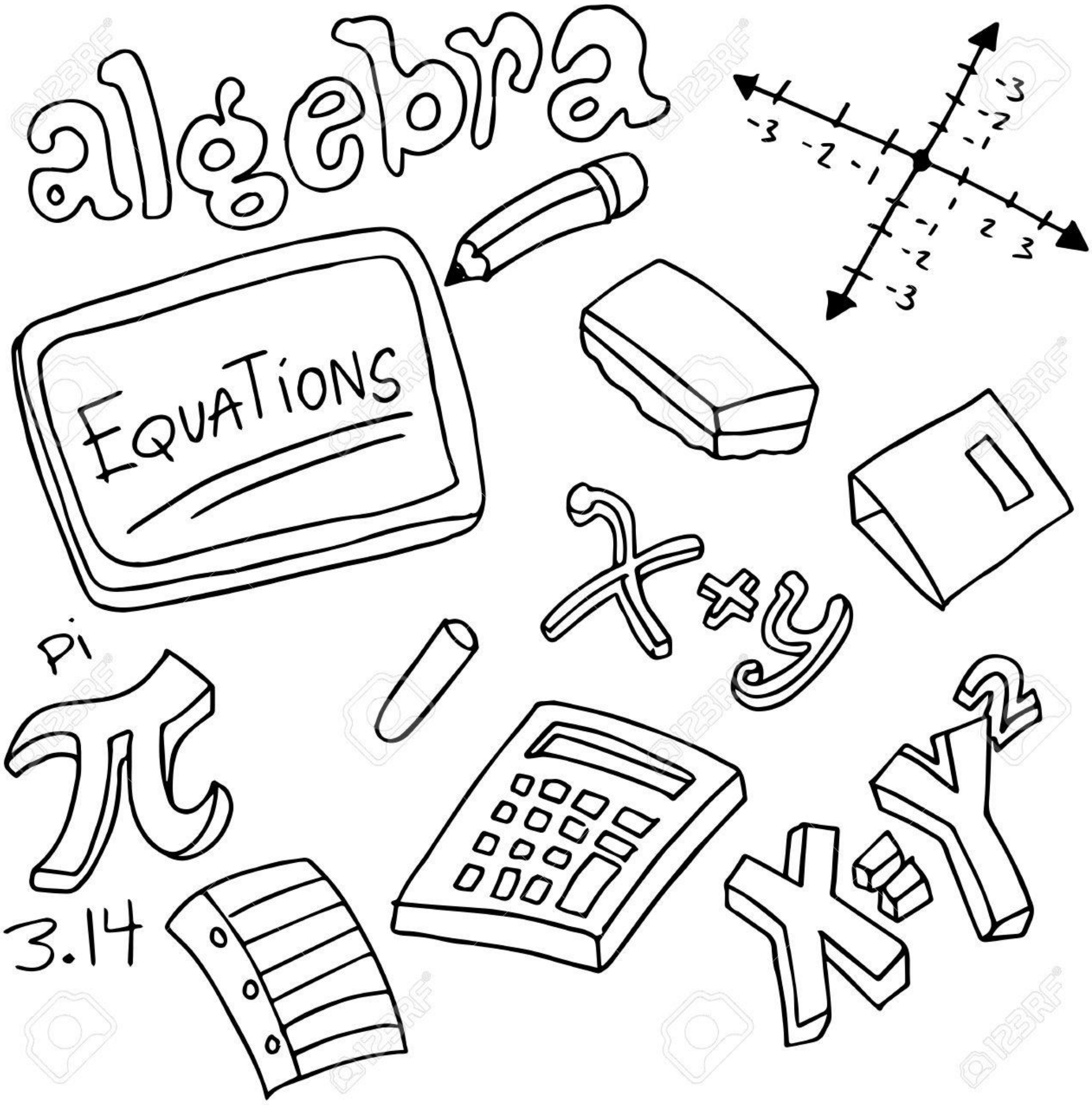
A-Level Further Mathematics



Bridging Course – Week 1

Entry Requirements for Studying A-level Further Mathematics?

- Students who are expected to achieve at least a grade 7 in GCSE Mathematics.
- Students who have enjoyed their GCSE Mathematics course, and who are keen problem solvers.
- Students who enjoy spending time working through a series of logical steps to find a conclusion.
- Students who enjoy the challenge of working through a difficult problem, and have the resilience required to keep going if they find something hard.
- Students who enjoy discussing their thoughts and ideas with others in a group.

What to expect from A-level Further Mathematics.

Just as languages provide the building blocks and rules we need to communicate, mathematics uses its own language, made up of numbers, symbols and formulae, to explore the rules we need to measure or identify essential problems like distance, speed, time, space, change, force and quantities. Studying maths helps us find patterns and structure in our lives. Practically, maths helps us put a price on things, create graphics, build websites, build skyscrapers and generally understand how things work or predict how they might change over time and under different conditions. As a subject, maths is continually growing and changing, as mathematician and scientists expand on what they already know to discover new theories and inventions.

A-level Further Mathematics is a demanding course. However, if you are a resilient and determined problem solver, it can also be extremely rewarding. Students should expect to complete regular pieces of written and online homework, as well as substantial amounts of independent practise, in order to fully embed concepts in the mind and apply them to a range of different situations. Communication of mathematics is as important as finding the correct answers and this will be tested through written work and regular assessments. Engagement with class discussion is essential in order to develop these communication skills.

Whilst A-Level Mathematics is a fascinating course to study in itself, Further Mathematics provides a great opportunity for enthusiastic mathematicians to broaden and deepen their subject knowledge. If you plan to apply for a STEM (Science, Technology, Engineering and Mathematics) degree you should consider taking Further Mathematics, it is also a fantastic qualification for those students who love maths and want to devote more time to the studying wider aspects of the subject.

This bridging course will provide you with a mixture of information about A-level Further Mathematics, and what to expect from the course, as well as key work to complete. Students who are expecting to study Further Mathematics at A-level, and are likely to meet the entry requirements, must complete the bridging course fully and thoroughly, to the best of their ability. You should complete all work on paper and keep it in a file, in an ordered way. You will submit it to your teacher in September. All of the work will be reviewed and selected work will be assessed, and you will be given feedback on it. This work will be signalled to you. If you do not have access to the internet, please contact the school and appropriate resources will be sent to you.

If you are thinking about studying Further Mathematics at A-level you should attempt this work to see whether or not you think studying a subject like this is right for you. If you later decide to study Further Mathematics, you must ensure you complete this work in full. This work should be completed after you have read and finished the Study Skills work that all of Year 12 should complete.

Course outline

Papers 1 & 2 – Mandatory Pure Core worth 50%	Paper 3 – Optional unit Mechanics worth 25%	Paper 4 – Optional unit Additional Pure worth 25%
• Two written exams, each lasting 90 minutes. • A calculator is allowed. • Assessment of content from the whole of the Pure Core and all of the Overarching Themes.	• A written exam lasting 90 minutes. • A calculator is allowed. • Assessment of the relevant content area and all of the Overarching Themes.	• A written exam lasting 90 minutes. • A calculator is allowed. • Assessment of the relevant content area and all of the Overarching Themes.

There is no controlled assessment (coursework) for A-Level Further Mathematics; your grade is based solely on the four exams detailed above.

Throughout year 12 and year 13, you will be assessed regularly through written homework tasks and in-class assessments. These will give you and your teachers opportunity to be continually reviewing your areas of strengths and weakness and give you a focus for your independent study.

At the end of year 12 and throughout year 13 you will be given exam papers to ensure you have plenty of practise answering exam questions and developing exam technique.

The following work requires a lot of time in order to ensure techniques are properly embedded and ready to be developed and built upon. Remember to **take regular breaks**, complete as much or as little additional practise as you need to successfully complete the section test for each section and **try your best**. There are links to instructional videos included, as well as notes and examples to help you with each topic area. Your maths teacher will review each of these topics during year 12.

<u>Matrix Arithmetic</u>

During the first week of this bridging course you will be developing skills in matrix arithmetic to include; dimensions of a matrix, addition and subtraction of matrices, multiplying a matrix by a scalar, multiplying a matrix by a matrix and the identity matrix.

You may wish to approach the work in the following way:

1. Read the notes and example pages and watch the video tutorials, making your own notes to file and keep
2. Complete as many of the questions from the two question sheets as you need to feel confident with the concepts
3. Complete and mark the review questions to assess your understanding
4. Return to step 1 if necessary

Helpful video tutorials for this topic:

<u>Hegarty Maths</u>: Videos and quizzes: 928 - 933
<u>Corbett Maths</u> : https://corbettmaths.com/2019/06/25/multiplying-by-a-scalar/ https://corbettmaths.com/2019/04/24/multiplying-matrices-2x2-by-2x1/ https://corbettmaths.com/2019/04/24/multiplying-matrices-2x2-by-2x2/ https://corbettmaths.com/2019/06/24/identity-matrix/
<u>TL Maths</u> This link has further videos for this section: https://sites.google.com/site/tlmaths314/home/a-level-further-maths-2017/full-a-level-videos/c-matrices/01-introducing-matrices C1-01 TO C1-13

1. Read:
   a. Matrix Arithmetic 1 and Matrix Arithmetic Further Power Point

2. Complete
   a. Matrix Arithmetic Questions 1 and Additional Matrix Arithmetic Questions

   b. Answers for review are at the back of that booklet.

3. Complete Review Questions

Section 1: Matrix arithmetic

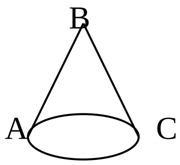
Notes and Examples

These notes contain subsections on

- Introducing matrices
- Multiplying a matrix by a scalar
- Multiplying matrices
- The identity matrix

Introducing matrices

A matrix is simply a way of storing information. For example, the diagram below shows a map of the roads linking three towns A, B and C. The corresponding '*direct route*' matrix is shown beside it.



$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \end{array}$$

In this section you learn to multiply a matrix by a number and to multiply two matrices.

Matrices are classified by number of rows and the number of columns they have. The matrix above has 3 rows and 3 columns, it is a 3×3 matrix (read as '3 by 3').

A matrix with m rows and n columns is an $m\times n$ matrix. This is called the **order** of the matrix.

A **square matrix** is a matrix with the same number of rows as columns.

Multiplying a matrix by a scalar

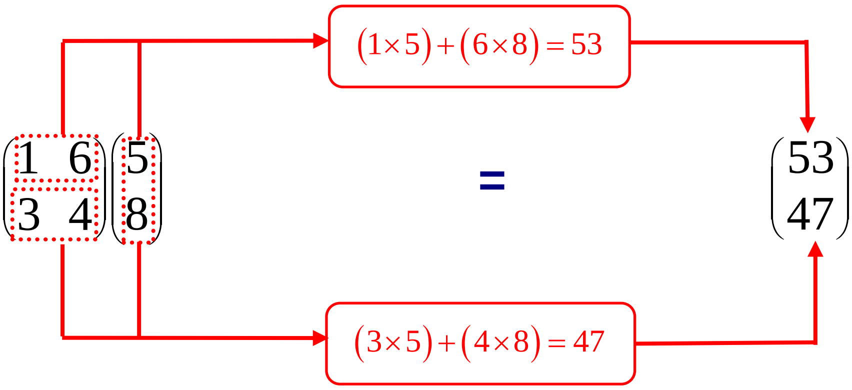
A matrix can be multiplied by a scalar (a number). Each element of the matrix is multiplied by the scalar.

For example, $\mathbf{A}=\begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix} \Rightarrow 3\mathbf{A}=\begin{pmatrix} 6 & -3 \\ 9 & 0 \end{pmatrix}$

Multiplying matrices

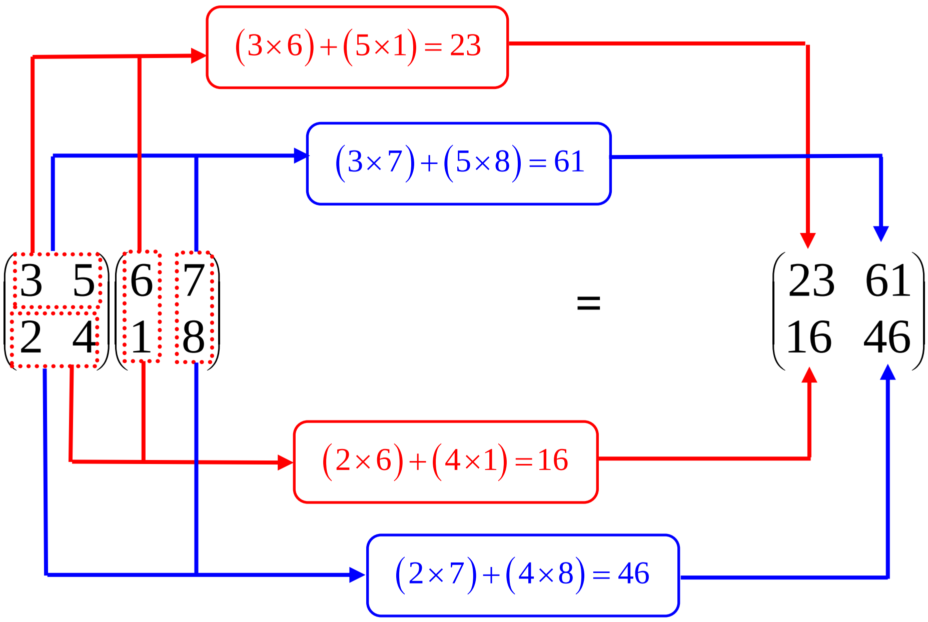
Multiplying matrices is an important skill which you must master. It takes a bit of getting used to, but after plenty of practice you will find it quite straightforward.

The diagram below shows the process of multiplying a 2 × 2 matrix by a 2 × 1 matrix.



So $\begin{pmatrix}1 & 6\\3 & 4\end{pmatrix}\begin{pmatrix}5\\8\end{pmatrix} = \begin{pmatrix}53\\47\end{pmatrix}$

The diagram below shows the steps used when multiplying a 2 × 2 matrix by another 2 × 2 matrix.



So $\begin{pmatrix}3 & 5\\2 & 4\end{pmatrix}\begin{pmatrix}6 & 7\\1 & 8\end{pmatrix} = \begin{pmatrix}23 & 61\\16 & 46\end{pmatrix}$

A similar technique applies to all matrix multiplications. You use each row of the first (i.e. left) matrix with each column, in turn, of the second matrix.

The important points to remember are:

- Use each row of the first matrix with each column of the second.
- When you are using row a of the first matrix with column b of the second matrix, the result gives you the element in row a, column b of the product matrix.
- To multiply matrices, the number of columns in the first matrix must be the same as the number of rows in the second matrix. If this is not the case, the matrices do not conform and cannot be multiplied.

You only need to be able to multiply a 2×2 matrix by another 2×2 matrix or by a 2×1 matrix.

Example 1

A is the matrix $\begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix}$.

B is the matrix $\begin{pmatrix} -3 & 0 \\ 1 & 4 \end{pmatrix}$.

C is the matrix $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

Find:

(i) **AB** (ii) **BA**

(iii) **AC** (iv) **BC**

Solution

(i) $$\mathbf{AB} = \begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} -3 & 0 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 2\times -3+3\times 1 & 2\times 0+3\times 4 \\ -1\times -3+5\times 1 & -1\times 0+5\times 4 \end{pmatrix} = \begin{pmatrix} -3 & 12 \\ 8 & 20 \end{pmatrix}$$

(ii) $$\mathbf{BA} = \begin{pmatrix} -3 & 0 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} -3\times 2+0\times -1 & -3\times 3+0\times 5 \\ 1\times 2+4\times -1 & 1\times 3+4\times 5 \end{pmatrix} = \begin{pmatrix} -6 & -9 \\ -2 & 23 \end{pmatrix}$$

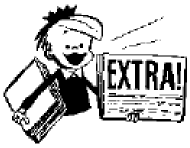
(iii) $$\mathbf{AC} = \begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 2\times 3+3\times -2 \\ -1\times 3+5\times -2 \end{pmatrix} = \begin{pmatrix} 0 \\ -13 \end{pmatrix}$$

(iv) $\mathbf{BC} = \begin{pmatrix} -3 & 0 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -3\times 3 + 0\times -2 \\ 1\times 3 + 4\times -2 \end{pmatrix}$

$= \begin{pmatrix} -9 \\ -5 \end{pmatrix}$

Note that it is not necessary to write out the calculations in full, as in the example above. It is shown here so that you can see what is being done. You may like to write it out in full until you feel confident, or you may feel able to miss out that step from the beginning.

Notice the important point that matrix multiplication, unlike the multiplication of numbers, is not commutative: i.e. $\mathbf{AB} \neq \mathbf{BA}$.



In addition, matrix multiplication has the following properties:

- Matrix multiplication is associative
- Matrix multiplication is distributive

The identity matrix

The matrix $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called the 2×2 **identity matrix** because when you multiply any 2×2 matrix **A** by **I** you get **A** as the answer.
I acts like the number 1 in the multiplication of numbers.

This means that for any 2×2 matrix **A**: $\mathbf{IA} = \mathbf{AI} = \mathbf{A}$.

Section 1: Matrix arithmetic

Crucial points

1. **Check your answers carefully**
   It's easy to make careless mistakes in matrix multiplication.

2. **Make sure that you can do matrix multiplication confidently**
   This will also be needed in Section 2.

3. **Remember that matrix multiplication is not commutative**
   This means that **AB** ≠ **BA**. This is an easy mistake to make as we are all used to ordinary multiplication being commutative.

4. **Make sure you know the significance of the identity matrix**
   The identity matrix, **I** = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, behaves in a similar way to the 1 in the multiplication of numbers: i.e. for any square matrix **A**, **AI** = **IA** = **A**.

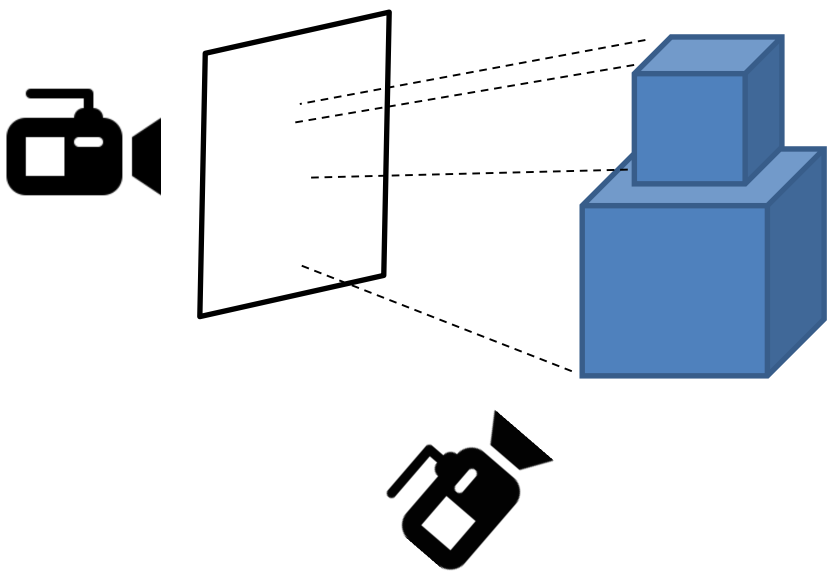
Introduction

A matrix (plural: matrices) is **simply an 'array' of numbers**, e.g.

$$\begin{pmatrix} 1 & 3 & -7 \\ 4 & 0 & 5 \end{pmatrix}$$

On a simple level, a matrix is simply a way to organise values into rows and columns, and represent these multiple values as a single structure.

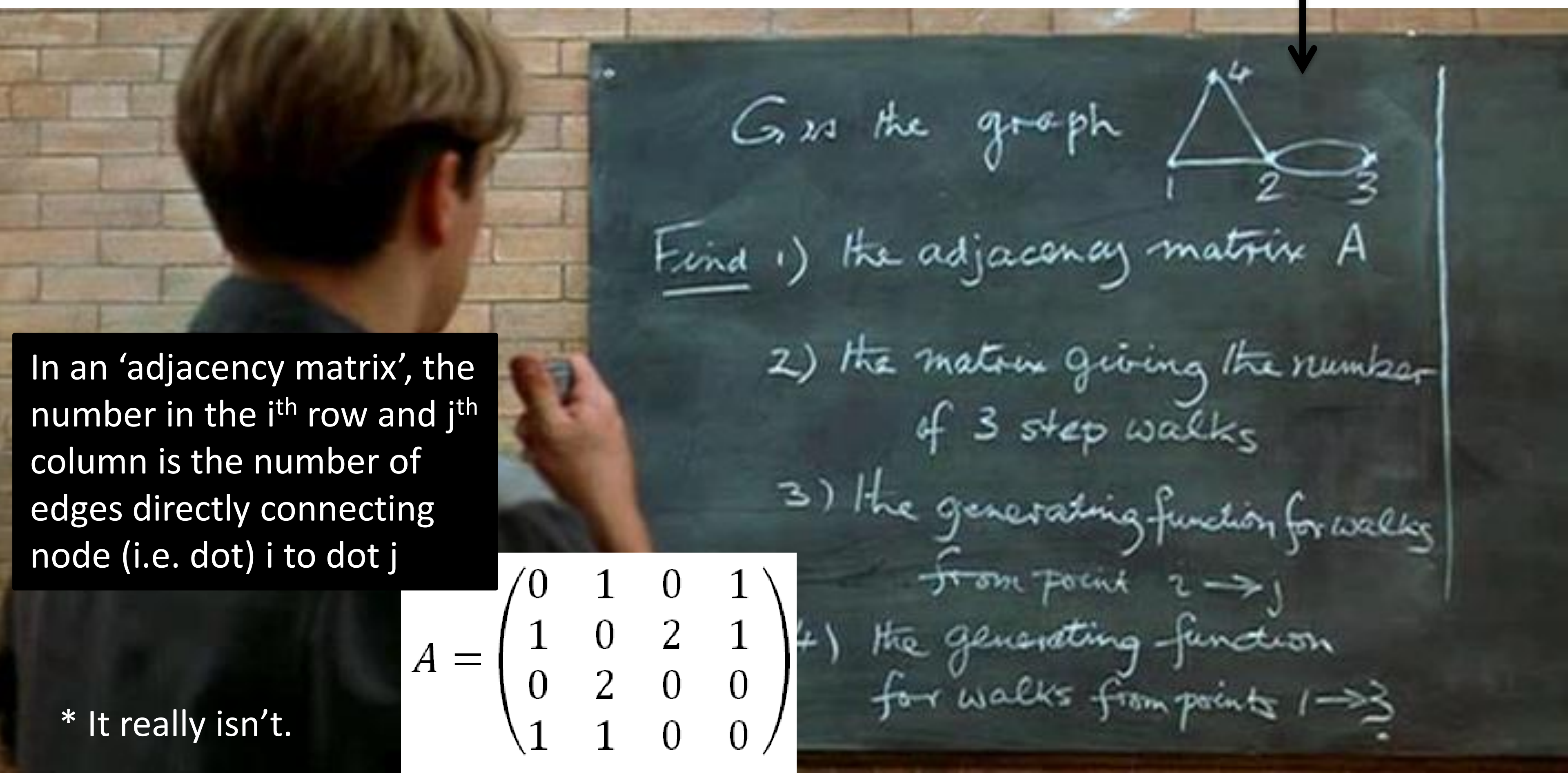
For the purposes of Further Maths, you should understand matrices as **a way to transform points**.



Matrices are particularly useful in 3D graphics, as matrices can be used to carry out rotations/enlargements (useful for changing the camera angle) or project into a 2D 'viewing' plane.

(Just for Fun) Using matrices to represent data

This is a scene from the film **Good Will Hunting**.
Maths professor Lambeau poses a “difficult”* problem for his graduate students from algebraic graph theory, the first part asking for a matrix representation of this graph.
Matt Damon anonymously solves the problem while on a cleaning shift.



In an ‘adjacency matrix’, the number in the i^{th} row and j^{th} column is the number of edges directly connecting node (i.e. dot) i to dot j

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

* It really isn’t.

Using matrices to represent data

Method 2: Iterative Clique Merging

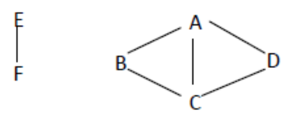
An alternative approach proposed here generates a number of seeds, where each seed is an island of closely connected features, before merging them into the required number of clusters.

Instead of normalising the columns in M as before, we scale all elements down so that the maximum element is 1. We define a threshold α ($0 < \alpha \leq 1$) such that if $M_{ij} \geq \alpha$, the bond between features i and j is considered to be 'strong', and 'weak' otherwise. This allows us to construct an undirected graph G, where we define the nodes to be the features, and the edges to be the 'strong bonds'.

To demonstrate this, take an example such that the co-occurrence matrix M is:

	A	B	C	D	E	F
A	1	1	1	1	0	0
B	1	1	1	0	0	0
C	1	1	1	1	0	0
D	1	0	1	1	0	0
E	0	0	0	0	1	1
F	0	0	0	0	1	1

with features A to F. The value of α is arbitrary here since all non-zero values are 1. This produces the graph:



[4] More specifically, the function is *f(x) = log(x+1)*. This ensures that 0 maps to 0, rather than produce an error.
[5] The centroid of a cluster is the mean of its constituent vectors.

A worked example

Suppose a mark scheme consisted of 2 marks, with the following criteria:

1) Mention of a 'dog'.
2) Mention of the act of 'walking' or 'exercise'.

Suppose also the following training answers were supplied, their classification denoted in brackets:

A. The big dog started barking. (1)
B. The dog buried his bone. (1)
C. Adam enjoys walking and exercise. (1)
D. The dog enjoyed exercising. (2)
E. The big lecturer scared the timid student. (0)

Each of the answers would be semantically analysed and deconstructed into its constituent features. For the sake of simplicity, we extract only 4: 'dog', 'big', 'walk' and 'exercise'. We could therefore generate the following incidence matrix A for the above answers:

	dog	big	walk	exercise
A	1	1	0	0
B	1	0	0	0
C	0	0	1	1
D	1	0	0	1
E	0	1	0	0

Computing $M = A^T . A$, taking the log of each value, and normalising the columns, we obtain:

	dog	big	walk	exercise
dog	0.5	0.387	0	0.279
big	1	0.613	0	0
walk	0	0	0.5	0.279
exercise	1	0	0.5	0.442

By inspection of the non-zero values, it is clear that the two groups obtained by clustering are G_1 = { dog, big } and G_2 = { walk, exercise }. This is consistent with the original mark scheme. We create a standard classifier for each of these groups (say C_1 and C_2), and now use the rows of the matrix A to

Matrix Algebra

Matrix Fundamentals

Understand the dimensions of a matrix, and operations on matrices, such as addition, scalar multiplication and matrix multiplication.

Matrix Fundamentals

#1 Dimensions of Matrices

The dimension of a matrix is its size, in terms of its number of rows and columns.

Matrix	Dimensions
$\begin{pmatrix} 1 & 3 & -7 \\ 4 & 0 & 5 \end{pmatrix}$	2×3
$\begin{pmatrix} 1 \\ 6 \\ -3 \end{pmatrix}$	?
$\begin{pmatrix} 1 & 6 & 0 \end{pmatrix}$	?

Matrix Fundamentals

#1 Dimensions of Matrices

The dimension of a matrix is its size, in terms of its number of rows and columns.

Matrix	Dimensions
$\begin{pmatrix} 1 & 3 & -7 \\ 4 & 0 & 5 \end{pmatrix}$	2×3
$\begin{pmatrix} 1 \\ 6 \\ -3 \end{pmatrix}$	3×1
$\begin{pmatrix} 1 & 6 & 0 \end{pmatrix}$	1×3

#2 Notation/Names for Matrices

A matrix can have square or curvy brackets*.

$$\begin{pmatrix} 7 & 1 & 2 \\ 6 & 1 & 5 \end{pmatrix} \qquad \begin{bmatrix} 1 \\ 6 \\ -3 \end{bmatrix} \qquad \begin{pmatrix} 1 & 6 & 0 \end{pmatrix}$$

Matrix

Column Vector
(The vector you know and love)

Row Vector

So a matrix with one column is simply a vector in the usual sense.

* The textbook only uses curvy.

#3 Variables for Matrices

If we wish a variable to represent a matrix, we use bold, capital letters.

$$\boldsymbol{A} = \begin{pmatrix} 1 \\ 6 \\ -3 \end{pmatrix}$$

$$\boldsymbol{C} = \boldsymbol{P}^{2}\boldsymbol{T}\boldsymbol{P}$$

#4 Adding/Subtracting Matrices

Simply add/subtract the corresponding elements of each matrix.
They must be of the same dimension.

$$\begin{pmatrix} 1 & 3 & -7 \\ 4 & 0 & 5 \end{pmatrix} + \begin{pmatrix} 6 & -2 & 9 \\ 2 & 1 & 0 \end{pmatrix} = \boxed{?}$$

$$\begin{pmatrix} 3 & 0 \\ -1 & 2 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} q & -3 \\ 1 & 1 \\ -4 & 1 \end{pmatrix} = \boxed{?}$$

Matrix Fundamentals

#4 Adding/Subtracting Matrices

Simply add/subtract the corresponding elements of each matrix.
They must be of the same dimension.

$$\begin{pmatrix} 1 & 3 & -7 \\ 4 & 0 & 5 \end{pmatrix} + \begin{pmatrix} 6 & -2 & 9 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 7 & 1 & 2 \\ 6 & 1 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 \\ -1 & 2 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} q & -3 \\ 1 & 1 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} 3-q & 3 \\ -2 & 1 \\ 4 & 2 \end{pmatrix}$$

#5 Scalar Multiplication

A scalar is a number which can 'scale' the elements inside a matrix/vector.

1

$$3\begin{pmatrix} 1 & 3 & -7 \\ 4 & 0 & 5 \end{pmatrix} = \quad ?$$

2

$$\boldsymbol{A} = \begin{pmatrix} q & -3 \\ 1 & 1 \\ -4 & 1 \end{pmatrix} \qquad 2\boldsymbol{A} = \quad ?$$

3

$$\begin{pmatrix} -3 \\ k \end{pmatrix} + k\begin{pmatrix} 2k \\ 2k \end{pmatrix} = \begin{pmatrix} k \\ 6 \end{pmatrix} \qquad k = \quad ?$$

#5 Scalar Multiplication

A scalar is a number which can ‘scale’ the elements inside a matrix/vector.

1

$$3\begin{pmatrix} 1 & 3 & -7 \\ 4 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 9 & -21 \\ 12 & 0 & 15 \end{pmatrix}$$

2

$$\boldsymbol{A} = \begin{pmatrix} q & -3 \\ 1 & 1 \\ -4 & 1 \end{pmatrix} \qquad 2\boldsymbol{A} = \begin{pmatrix} 2q & -6 \\ 2 & 2 \\ -8 & 2 \end{pmatrix}$$

3

$$\begin{pmatrix} -3 \\ k \end{pmatrix} + k\begin{pmatrix} 2k \\ 2k \end{pmatrix} = \begin{pmatrix} k \\ 6 \end{pmatrix} \qquad k = \frac{3}{2}$$

#6 Matrix Multiplication

This is where things get slightly more complicated...

Now repeat for the next row of the left matrix...

$$\begin{bmatrix} 1 & 0 & 3 & -2 \\ 2 & 8 & 4 & 3 \\ 7 & -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 1 & 7 \\ 0 & 3 \\ 8 & -3 \end{bmatrix} \qquad \begin{bmatrix} -11 & 16 \\ 42 & 61 \\ 50 & -6 \end{bmatrix}$$

We start with this row and column, and sum the products of each pair.
(1 x 5) + (0 x 1) + (3 x 0) + (-2 x 8) = -11

Further Example

June 2012 Paper 1 Q2

$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$

Work out the matrix **AB**.

$$= \begin{pmatrix} 10 \\ 17 \end{pmatrix}$$

Test Your Understanding

Now you have a go...

a If $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}, AB =$?

b $\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} =$?

c $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^2 =$?

☠ $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^k =$?

☠ $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} =$?

☠ $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} =$?

Bro Exam Note: In IGCSEFM, you will only have to multiply either a 2×2 by 2×1 or 2×2 by 2×1.

Test Your Understanding

Now you have a go...

a If $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}, AB = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{3} & \mathbf{3} \end{pmatrix}$

b $\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} \mathbf{2} \\ \mathbf{6} \end{pmatrix}$

c $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^2 = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$

☠ $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & ak \\ 0 & 1 \end{pmatrix}$

☠ $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 14 \end{pmatrix}$

☠ $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$

Identity Matrix

Let $\boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
Determine:

$$AI = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix}$$
$$IA = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix}$$

✎ $\boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is known as the ‘identity matrix’.
Multiplying by it has no effect, i.e. $AI = IA = A$ for any matrix A.

It may seem pointless to have such a matrix, but it’ll have more importance when we consider matrices as ‘transformations’ later. Although admittedly you won’t quite fully appreciate why we have it unless you do Further Maths A Level…

Exercise 1

1 Work out

(a) $\begin{pmatrix} 4 & 2 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} 7 \\ 1 \end{pmatrix}$?

(b) $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} -3 \\ -4 \end{pmatrix}$?

(c) $2\begin{pmatrix} 5 & -2 \\ 6 & -3 \end{pmatrix}$?

(d) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix}$?

(e) $6\begin{pmatrix} -4 & 7 \\ -1 & -3 \end{pmatrix}$?

(f) $\begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} -3 \\ 6 \end{pmatrix}$?

2

$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 7 & 4 \\ 5 & 3 \end{pmatrix}$ $\mathbf{C} = \begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix}$

Work out

(a) **AB** ?

(b) **BC** ?

(c) 3**A** ?

(d) **BA** ?

(e) –**C** ?

(f) $\mathbf{B}\begin{pmatrix} 1 & -4 \\ -5 & 7 \end{pmatrix}$?

3

$\mathbf{P} = \begin{pmatrix} -2 & 0 \\ 5 & 1 \end{pmatrix}$ $\mathbf{Q} = \begin{pmatrix} -4 & 1 \\ 3 & -2 \end{pmatrix}$ $\mathbf{C} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

Work out

(a) $\mathbf{P}^2$?

(b) **QP** ?

(c) 5**Q** ?

(d) **PC** ?

(e) **IQ** ?

(f) 3**I** ?

Exercise 1

4

(a) $\begin{pmatrix}2 & -1\\1 & 3\end{pmatrix}\begin{pmatrix}0 & 3\\1 & -4\end{pmatrix}$?

(b) $\begin{pmatrix}-3 & -2\\-1 & 5\end{pmatrix}\begin{pmatrix}-2 & 4\\3 & 4\end{pmatrix}$?

(c) $\begin{pmatrix}3 & 2\\7 & 5\end{pmatrix}\begin{pmatrix}5 & -2\\-7 & 3\end{pmatrix}$?

(d) $\begin{pmatrix}10 & -7\\9 & 8\end{pmatrix}\begin{pmatrix}2 & 4\\-2 & 3\end{pmatrix}$?

(e) $\begin{pmatrix}1 & -2\\3 & -5\end{pmatrix}\begin{pmatrix}2 & 3\\1 & 4\end{pmatrix}$?

(f) $\begin{pmatrix}2 & 3\\1 & 4\end{pmatrix}\begin{pmatrix}1 & -2\\3 & -5\end{pmatrix}$?

5 Work out, giving your answers as simply as possible.

(a) $\begin{pmatrix}\sqrt{2} & 1\\-1 & 3\sqrt{2}\end{pmatrix}\begin{pmatrix}\sqrt{2} & 0\\-3 & -2\sqrt{2}\end{pmatrix}$?

(b) $\begin{pmatrix}-\frac{1}{2} & -1\\\frac{3}{2} & 5\end{pmatrix}\begin{pmatrix}-2 & 4\\-\frac{1}{2} & 3\end{pmatrix}$?

(c) $\begin{pmatrix}3 & 2\\7 & 5\end{pmatrix}^2$?

(d) $\begin{pmatrix}3\sqrt{3} & -4\\2 & 3\sqrt{3}\end{pmatrix}\begin{pmatrix}\sqrt{3} & 1\\-4 & 0\end{pmatrix}$?

(e) $\begin{pmatrix}\frac{1}{3} & \frac{1}{2}\\\frac{2}{3} & \frac{1}{4}\end{pmatrix}\begin{pmatrix}2 & 3\\1 & 4\end{pmatrix}$?

(f) $\begin{pmatrix}\sqrt{2} & 2\\7 & \sqrt{3}\end{pmatrix}^2$?

6 Work out, giving your answers as simply as possible.

(a) $\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}p\\p+1\end{pmatrix}$?

(b) $\begin{pmatrix}3 & 0\\0 & 3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$?

(c) $\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}m\\2m\end{pmatrix}$?

(d) $\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}\begin{pmatrix}-a & 0\\0 & a\end{pmatrix}$?

(e) $\begin{pmatrix}4t & 0\\0 & 4t\end{pmatrix}\begin{pmatrix}3 & 0\\0 & 3\end{pmatrix}$?

(f) $\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}3\\-2\end{pmatrix}$?

Exercise 1

7

Work out, giving your answers as simply as possible.

(a) $\begin{pmatrix} 2x & -3 \\ -5 & 4x \end{pmatrix}\begin{pmatrix} x & 3x \\ -3 & 0 \end{pmatrix}$?

(b) $\begin{pmatrix} a & 3a \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 7 & 8 \\ -10 & 11 \end{pmatrix}$?

(c) $\begin{pmatrix} x & 0 \\ 1 & x \end{pmatrix}^2$?

(d) $\begin{pmatrix} y & y \\ -3 & x \end{pmatrix}\begin{pmatrix} 2 & 3y \\ 0 & 1 \end{pmatrix}$?

(e) $\begin{pmatrix} a+1 & a \\ a+2 & a+1 \end{pmatrix}\begin{pmatrix} a+1 & -a \\ -a-2 & a+1 \end{pmatrix}$?

(f) $\begin{pmatrix} 3x & -3 \\ -9 & x+1 \end{pmatrix}^2$?

Exercise 1

1 Work out

(a) $\begin{pmatrix} 4 & 2 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} 7 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 30 \\ -16 \end{pmatrix}$

(b) $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} -3 \\ -4 \end{pmatrix}$ $\begin{pmatrix} -15 \\ -20 \end{pmatrix}$

(c) $2\begin{pmatrix} 5 & -2 \\ 6 & -3 \end{pmatrix}$ $\begin{pmatrix} 10 & -4 \\ 12 & -6 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

(e) $6\begin{pmatrix} -4 & 7 \\ -1 & -3 \end{pmatrix}$ $\begin{pmatrix} -24 & 42 \\ -6 & -18 \end{pmatrix}$

(f) $\begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} -3 \\ 6 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

2 $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 7 & 4 \\ 5 & 3 \end{pmatrix}$ $\mathbf{C} = \begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix}$

Work out

(a) **AB** $\begin{pmatrix} 9 & 5 \\ 41 & 24 \end{pmatrix}$

(b) **BC** $\begin{pmatrix} -10 & 17 \\ -7 & 12 \end{pmatrix}$

(c) 3**A** $\begin{pmatrix} 6 & -3 \\ 9 & 12 \end{pmatrix}$

(d) **BA** $\begin{pmatrix} 26 & 9 \\ 19 & 7 \end{pmatrix}$

(e) –**C** $\begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix}$

(f) $\mathbf{B}\begin{pmatrix} 1 & -4 \\ -5 & 7 \end{pmatrix}$ $\begin{pmatrix} -13 & 0 \\ -10 & 1 \end{pmatrix}$

3 $\mathbf{P} = \begin{pmatrix} -2 & 0 \\ 5 & 1 \end{pmatrix}$ $\mathbf{Q} = \begin{pmatrix} -4 & 1 \\ 3 & -2 \end{pmatrix}$ $\mathbf{C} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

Work out

(a) $\mathbf{P}^2$ $\begin{pmatrix} 4 & 0 \\ -5 & 1 \end{pmatrix}$

(b) **QP** $\begin{pmatrix} 13 & 1 \\ -16 & -2 \end{pmatrix}$

(c) 5**Q** $\begin{pmatrix} -20 & 5 \\ 15 & -10 \end{pmatrix}$

(d) **PC** $\begin{pmatrix} -6 \\ 13 \end{pmatrix}$

(e) **IQ** $\begin{pmatrix} -4 & 1 \\ 3 & -2 \end{pmatrix}$

(f) 3**I** $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

Exercise 1

4

(a) $\begin{pmatrix}2 & -1\\1 & 3\end{pmatrix}\begin{pmatrix}0 & 3\\1 & -4\end{pmatrix}$ $\begin{pmatrix}-1 & 10\\3 & -9\end{pmatrix}$

(b) $\begin{pmatrix}-3 & -2\\-1 & 5\end{pmatrix}\begin{pmatrix}-2 & 4\\3 & 4\end{pmatrix}$ $\begin{pmatrix}0 & -20\\17 & 16\end{pmatrix}$

(c) $\begin{pmatrix}3 & 2\\7 & 5\end{pmatrix}\begin{pmatrix}5 & -2\\-7 & 3\end{pmatrix}$ $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

(d) $\begin{pmatrix}10 & -7\\9 & 8\end{pmatrix}\begin{pmatrix}2 & 4\\-2 & 3\end{pmatrix}$ $\begin{pmatrix}34 & 19\\2 & 60\end{pmatrix}$

(e) $\begin{pmatrix}1 & -2\\3 & -5\end{pmatrix}\begin{pmatrix}2 & 3\\1 & 4\end{pmatrix}$ $\begin{pmatrix}0 & -5\\1 & -11\end{pmatrix}$

(f) $\begin{pmatrix}2 & 3\\1 & 4\end{pmatrix}\begin{pmatrix}1 & -2\\3 & -5\end{pmatrix}$ $\begin{pmatrix}11 & -19\\13 & -22\end{pmatrix}$

5 Work out, giving your answers as simply as possible.

(a) $\begin{pmatrix}\sqrt{2} & 1\\-1 & 3\sqrt{2}\end{pmatrix}\begin{pmatrix}\sqrt{2} & 0\\-3 & -2\sqrt{2}\end{pmatrix}$ $\begin{pmatrix}-1 & -2\sqrt{2}\\-10\sqrt{2} & -12\end{pmatrix}$

(b) $\begin{pmatrix}-\frac{1}{2} & -1\\\frac{3}{2} & 5\end{pmatrix}\begin{pmatrix}-2 & 4\\-\frac{1}{2} & 3\end{pmatrix}$ $\begin{pmatrix}\frac{3}{2} & -5\\-\frac{11}{2} & 21\end{pmatrix}$

(c) $\begin{pmatrix}3 & 2\\7 & 5\end{pmatrix}^2$ $\begin{pmatrix}23 & 16\\56 & 39\end{pmatrix}$

(d) $\begin{pmatrix}3\sqrt{3} & -4\\2 & 3\sqrt{3}\end{pmatrix}\begin{pmatrix}\sqrt{3} & 1\\-4 & 0\end{pmatrix}$ $\begin{pmatrix}25 & 3\sqrt{3}\\-10\sqrt{3} & 2\end{pmatrix}$

(e) $\begin{pmatrix}\frac{1}{3} & \frac{1}{2}\\\frac{2}{3} & \frac{1}{4}\end{pmatrix}\begin{pmatrix}2 & 3\\1 & 4\end{pmatrix}$ $\begin{pmatrix}\frac{7}{6} & 3\\\frac{19}{12} & 3\end{pmatrix}$

(f) $\begin{pmatrix}\sqrt{2} & 2\\7 & \sqrt{3}\end{pmatrix}^2$ $\begin{pmatrix}16 & 2\sqrt{2}+2\sqrt{3}\\7\sqrt{2}+7\sqrt{3} & 17\end{pmatrix}$

6 Work out, giving your answers as simply as possible.

(a) $\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}p\\p+1\end{pmatrix}$ $\begin{pmatrix}-p\\-p-1\end{pmatrix}$

(b) $\begin{pmatrix}3 & 0\\0 & 3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$ $\begin{pmatrix}3x\\3y\end{pmatrix}$

(c) $\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}m\\2m\end{pmatrix}$ $\begin{pmatrix}2m\\m\end{pmatrix}$

(d) $\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}\begin{pmatrix}-a & 0\\0 & a\end{pmatrix}$ $\begin{pmatrix}-2a & 0\\0 & 2a\end{pmatrix}$

(e) $\begin{pmatrix}4t & 0\\0 & 4t\end{pmatrix}\begin{pmatrix}3 & 0\\0 & 3\end{pmatrix}$ $\begin{pmatrix}12t & 0\\0 & 12t\end{pmatrix}$

(f) $\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}3\\-2\end{pmatrix}$ $\begin{pmatrix}-3\\-2\end{pmatrix}$

Exercise 1

7

Work out, giving your answers as simply as possible.

(a) $\begin{pmatrix} 2x & -3 \\ -5 & 4x \end{pmatrix}\begin{pmatrix} x & 3x \\ -3 & 0 \end{pmatrix}$

$\begin{pmatrix} 2x^2+9 & 6x^2 \\ -17x & -15x \end{pmatrix}$

(b) $\begin{pmatrix} a & 3a \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 7 & 8 \\ -10 & 11 \end{pmatrix}$

$\begin{pmatrix} -23a & 41a \\ -24 & -5 \end{pmatrix}$

(c) $\begin{pmatrix} x & 0 \\ 1 & x \end{pmatrix}^2$

$\begin{pmatrix} x^2 & 0 \\ 2x & x^2 \end{pmatrix}$

(d) $\begin{pmatrix} y & y \\ -3 & x \end{pmatrix}\begin{pmatrix} 2 & 3y \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 2y & 3y^2+y \\ -6 & -9y+x \end{pmatrix}$

(e) $\begin{pmatrix} a+1 & a \\ a+2 & a+1 \end{pmatrix}\begin{pmatrix} a+1 & -a \\ -a-2 & a+1 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(f) $\begin{pmatrix} 3x & -3 \\ -9 & x+1 \end{pmatrix}^2$

$\begin{pmatrix} 9x^2+27 & -12x-3 \\ -36x-9 & x^2+2x+28 \end{pmatrix}$

Harder Multiplication Questions

Matrix multiplications may give us **simultaneous equations**, which we solve in the usual way.

June 2013 Paper 2 Q12

Work out all solutions for x and y if $\begin{pmatrix} x & 3 \\ 1 & y \end{pmatrix}\begin{pmatrix} x \\ -4 \end{pmatrix} = \begin{pmatrix} 4x \\ 8 \end{pmatrix}$

$$\begin{pmatrix} x^2 - 12 \\ x - 4y \end{pmatrix} = \begin{pmatrix} 4x \\ 8 \end{pmatrix}$$

$$x^2 - 12 = 4x \qquad x^2 - 4x - 12 = 0$$

$$x = -2 \; or \; x = 6$$

$$y = -2.5 \; or \; y = -0.5$$

Test Your Understanding

AQA Worksheet 2

Work out the values of a, b and c.

$$\begin{pmatrix} 2 & a \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & b \end{pmatrix} = \begin{pmatrix} 12 & 26 \\ c & 13 \end{pmatrix}$$

$$2 + 2a = 12 \quad \rightarrow \quad a = 5$$

$$6 + ab = 26 \quad \rightarrow \quad b = 4$$

$$3 + 2 = c \quad \rightarrow \quad c = 5$$

Exercise 1b

1

$$\begin{pmatrix} -2 & a \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 22 \\ 9 \end{pmatrix}$$

?

Work out the value of a.

2 June 2013 Paper 2 Q11

(a) Work out $\begin{pmatrix} 2 & -1 \\ \frac{1}{3} & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ a & c \end{pmatrix}$

?

Give your answer in terms of a, b and c.

(b) You are given that $\begin{pmatrix} 2 & -1 \\ \frac{1}{3} & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ a & c \end{pmatrix} = \mathbf{I}$ where **I** is the identity matrix.

Work out the values of a, b and c.

?

3 Set 2 Paper 2 Q16

Matrix $\mathbf{P} = \begin{pmatrix} 2 & 3 \\ a & b \end{pmatrix}$ Matrix $\mathbf{Q} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

You are given that $\mathbf{PQ} = \mathbf{QP}$

Work out the values of a and b.

?

4 Set 4 Paper 1 Q17

$$\begin{pmatrix} 2 & a \\ 1 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Work out **all** possible pairs of values of a and b.

?

Exercise 1b

1

$$\begin{pmatrix} -2 & a \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 22 \\ 9 \end{pmatrix}$$

Work out the value of a.

$$-6 + 7a = 22$$
$$a = 4$$

2 June 2013 Paper 2 Q11

(a) Work out $\begin{pmatrix} 2 & -1 \\ \frac{1}{3} & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ a & c \end{pmatrix}$

$$\begin{pmatrix} -a & 2b - c \\ 0 & \frac{1}{3}b \end{pmatrix}$$

Give your answer in terms of a, b and c.

(b) You are given that $\begin{pmatrix} 2 & -1 \\ \frac{1}{3} & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ a & c \end{pmatrix} = \mathbf{I}$ where **I** is the identity matrix.

Work out the values of a, b and c.

$$a = -1, b = 3, c = 6$$

3 Set 2 Paper 2 Q16

Matrix $\mathbf{P} = \begin{pmatrix} 2 & 3 \\ a & b \end{pmatrix}$ Matrix $\mathbf{Q} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

You are given that $\mathbf{PQ} = \mathbf{QP}$

Work out the values of a and b.

$$PQ = \begin{pmatrix} 2 & 5 \\ a & a + b \end{pmatrix}$$
$$QP = \begin{pmatrix} 2 + a & 3 + b \\ a & b \end{pmatrix}$$
$$\therefore a = 0, b = 2$$

4 Set 4 Paper 1 Q17

$$\begin{pmatrix} 2 & a \\ 1 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Work out **all** possible pairs of values of a and b.

$$2a + ab = -1$$
$$a - 3b = 2$$

$$a = 2 + 3b$$
$$\therefore 2(2 + 3b) + (2 + 3b)b = -1$$
$$3b^2 + 8b + 5 = 0$$
$$(3b + 5)(b + 1) = 0$$
$$b = -1 \text{ or } b = -\frac{5}{3}$$
$$a = -1 \text{ or } a = -3$$