Numbers — Higher

Prime factor decomposition, HCF and LCM

Prime Factor Decomposition

Break numbers down into prime factors

Tree method and ladder method

$72 = 2 \times 2 \times 2 \times 3 \times 3$

$72 = 2^3 \times 3^2$

$84 = 2 \times 2 \times 3 \times 7$

$84 = 2^2 \times 3 \times 7$

HCF

Highest common factor

The largest number that divides into two or more numbers

Use long format of Prime Factor Decomposition

HCF of 48 and 180

$48 = 2 \times 2 \times 2 \times 2 \times 3$

$180 = 2 \times 2 \times 3 \times 3 \times 5$

$2 \times 2 \times 3 = 12$

HCF of 84 and 980

$84 = 2 \times 2 \times 3 \times 7$

$980 = 2 \times 2 \times 5 \times 7 \times 7$

$2 \times 2 \times 7 = 28$

LCM

Lowest common multiple

The smallest number that occurs in the times table of two or more numbers.

LCM of 6 and 45

$6 = 2 \times 3$

$45 = 3 \times 3 \times 5$

$2 \times 3 \times 3 \times 5 = 90$

LCM of 48 and 180

$48 = 2^4 \times 3$

$180 = 2^2 \times 3^2 \times 5$

$2^4 \times 3^2 \times 5 = 720$

- Divide by a prime
- Multiply Primes
- Write in index form
- Prime factor decomposition
- Identify shared factors
- Multiply values
- Multiply together all prime factors apart from duplicates
- In index form: Multiply Highest Power of each prime

Powers and Roots

Negative powers

Negative powers are fractions

5^{-2}	$1/25$
5^{-1}	$1/5$
5^{0}	1
5^{1}	5
5^{2}	$5 \times 5 = 25$
5^{3}	$5 \times 5 \times 5 = 125$

$\div 5$ (between each row)

A negative power means "Take the reciprocal and make the power positive"

$\left(\frac{4}{5}\right)^{-2} = \left(\frac{5}{4}\right)^{2} = \frac{5^2}{4^2} = \frac{25}{16}$

- Find Reciprocal
- Apply Positive Power
- Apply top and bottom

$x^{-n} \Rightarrow 1/x^n$

Fractional powers

Numerator and denominator are important

2-Stage Powers: $x^{\frac{m}{n}} = (\sqrt[n]{x})^m$

Root by denominator first

Then power of numerator

$16^{\frac{3}{2}} = (\sqrt{16})^3 = 64$

Negative Fractional Powers: Apply reciprocal first!

$\left(\frac{27}{64}\right)^{-\frac{2}{3}} = \left(\frac{64}{27}\right)^{\frac{2}{3}} = \left(\frac{4}{3}\right)^2 = \frac{16}{9}$

Multiplying and Dividing fractions

Multiplying

$\frac{2}{3} \times \frac{5}{7} = \frac{10}{21}$ — Multiply across the top and bottom

Check to see if you can cross cancel

$\frac{4}{15} \times \frac{3}{8} \Rightarrow \frac{\cancel{4}^{1}}{\cancel{15}_{5}} \times \frac{\cancel{3}^{1}}{\cancel{8}_{2}} \Rightarrow \frac{1}{5} \times \frac{1}{2} \Rightarrow \frac{1}{10}$

Convert if mixed numbers → Simplify vertically and diagonally → Multiply across top and bottom → Convert and simplify if required

Dividing

$\frac{10}{3} \div \frac{2}{3}$ — Multiply by the Reciprocal → $\frac{10}{3} \times \frac{3}{2}$

Finding the reciprocal of a fraction swaps the numerator and denominator

$\frac{4}{7} \div \frac{6}{5} \Rightarrow \frac{\cancel{4}^{2}}{7} \times \frac{5}{\cancel{6}_{3}} \Rightarrow \frac{10}{21}$

$2\frac{1}{5} \div 3\frac{3}{10} \Rightarrow \frac{11}{5} \div \frac{33}{10} \Rightarrow \frac{\cancel{11}^{1}}{\cancel{5}_{1}} \times \frac{\cancel{10}^{2}}{\cancel{33}_{3}} = \frac{2}{3}$

Convert mixed numbers → Find the reciprocal → Change sign → Simplify and Multiply

Units – Areas and Volume

Area and Volume

Area is a 2D measurement formed by multiplying two lengths

$100mm^2 = 1cm^2$

$10{,}000cm^2 = 1m^2$

$1{,}000{,}000m^2 = 1km^2$

Squared units results in squared conversion factors

Volume is a 3D measurement formed by multiplying three lengths

$1000mm^3 = 1cm^3$

$1{,}000{,}000cm^3 = 1m^3$

$1{,}000{,}000{,}000m^3 = 1km^3$

Cubed units have cubed conversion factors

Standard form

Basic Structure

$1 \le a < 10 \longleftarrow a \times 10^b \longrightarrow$ Whole number

$2.83 \times 10^6 = 2830000$

Positive power of 10 = Large number

$3.14 \times 10^{-4} = 0.000314$

Negative power of 10 = Small decimal number

Add/Subtract Standard form

Take numbers out of Standard form.

Add/Subtract values.

Convert answer back to Standard form.

$(3.23 \times 10^4) + (8.2 \times 10^3)$

$= 32300 + 8200$

$= 40500$

$= 4.05 \times 10^4$

Multiply/Divide Standard form

Separate the numbers and powers of 10.

Multiply/Divide numbers,

Apply laws of indices to power of 10s

Give answer in Standard form

$(4.6 \times 10^4)(3 \times 10^3)$

$4.6 \times 3 \times 10^4 \times 10^3$

13.8×10^7 ✗

1.38×10^8 ✓

$(1.56 \times 10^{-4}) \div (7.5 \times 10^{-7})$

$1.56 \div 7.5 \times 10^{-4} \div 10^{-7}$

0.208×10^3 ✗

2.08×10^2 ✓

Surds and Rationalising the denominator

Surds are expressions which contain an irrational square root

$\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$ — $\sqrt{3} \times \sqrt{7} = \sqrt{3 \times 7} = \sqrt{21}$

$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$ — $\frac{\sqrt{6}}{\sqrt{10}} = \sqrt{\frac{\cancel{6}^{3}}{\cancel{10}_{5}}} = \sqrt{\frac{3}{5}}$

$\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$ — $\sqrt{5} + \sqrt{20} = \sqrt{25}$ ✗

Writing in the form $a\sqrt{b}$

Think square numbers

Square Factors = 4, 25, 100

Choose the largest square factor

$\sqrt{200}$

$\sqrt{100} \times \sqrt{2} = 10\sqrt{2}$

Rationalising

Rationalising the denominator involves removing all of the roots from the bottom of a fraction.

$\frac{6}{\sqrt{3}} \Rightarrow \frac{6}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$ — Multiply top and bottom by irrational root — $\Rightarrow \frac{6\sqrt{3}}{\sqrt{9}} \Rightarrow \frac{6\sqrt{3}}{3}$

A more complex denominator

$\frac{5}{3+\sqrt{2}} \Rightarrow \frac{5}{3+\sqrt{2}} \times \frac{3-\sqrt{2}}{3-\sqrt{2}}$ — Multiply top and bottom by Conjugate (opposite root)

$= \frac{5(3-\sqrt{2})}{(3+\sqrt{2})(3-\sqrt{2})}$ — Expand and simplify

$= \frac{15-5\sqrt{2}}{9 - 3\sqrt{2} + 3\sqrt{2} - 2} = \frac{15-5\sqrt{2}}{7}$

Compound growth and decay

An amount is increased or decreased by a percentage

The process is repeated several times at each interval

The most efficient way to do this is using a Multiplier

The general formula for compound growth and decay: Amount = Principal $\times (1 \pm \text{Rate})^{\text{Years}}$ (Growth: +, Decay: −)

£4000 is invested at a rate of 5% p.a for three years. Calculate Final value of the investment after three years.

$£4000 \times 1.05^3 = £4630.50$

A car worth £15000 depreciates in value at a rate of 15% p.a. What is the depreciated value of the car after 4 years

$£15000 \times 0.85^4 = £7830.09$

To calculate other parts of the formula, you will need to change the subject

Percentage change

Percentage increase/decrease

Increase 70 by 15%

Calculate percentage of amount. Add on for increase. Subtract for decrease.

Method 1- Unitary method: $10\% = 7$, $5\% = 3.5 \Rightarrow 15\% = 10.5$

Method 2- Decimal method: $15\% = 0.15 \Rightarrow 0.15 \times 70 \Rightarrow 15\% = 10.5$

Method 3- Calculator method

Original Value —×Multiplier factor→ New Value; New Value —÷ Multiplier Factor→ Original Value

To find original amount, work backwards and divide by multiplier factor

Increase or Decrease?

Add % to 100 for increase | Subtract % from 100 for decrease

Convert to decimal (÷ 100)

This is your multiplier factor

Increase of 23% = ×1.23

Decrease of 42% = ×0.58

$\text{Percentage change} = \frac{\text{difference}}{\text{original amount}} \times 100$

Estimating, Bounds and Error intervals

Approximation

Estimates tell us the rough value of a calculation

$\frac{103.5 \times 1.92}{51.36} \approx \frac{100 \times 2}{50}$

Rounding off makes it easier to calculate

Round values to 1s.f. → Perform Calculation

$\frac{8.41 \times 3.2}{0.00216} \approx \frac{8 \times 3}{0.002} \Rightarrow \frac{24}{0.002} \Rightarrow \frac{24000}{2}$

$= 12000$

Error Intervals

By definition, a rounded number does not give us the exact value

Lower Bound — The minimum a value might be

Upper Bound — The maximum a value might be

Continuous Values (Decimal values)

Halve accuracy level → Add on for Upper bound / Subtract for Lower bound

$240m$ *to nearest* $10m$

$235m \longleftarrow 240m \longrightarrow 245m$

$235m \le x < 245m$

Discrete values (Whole values)

The number of people on a train is 400 to the nearest 100

$350 \longleftarrow 400 \longrightarrow 449$

Recurring decimals

What is a recurring decimal?

A decimal number that will after a certain point, repeat itself indefinitely.

Written with a little dot above the number/s

$0.\dot{6} \longrightarrow 0.6666666666666\ldots$

$0.21\dot{3} \longrightarrow 0.2133333333333\ldots$

$0.\dot{8}4\dot{1} \longrightarrow 0.841841841841\ldots$

A recurring decimal as a fraction

$0.\dot{x} \longrightarrow$ A single recurring digit will be a fraction over 9 — $\frac{x}{9}$

$0.\dot{x}\dot{y} \longrightarrow$ A double recurring digit will be a fraction over 99 — $\frac{xy}{99}$

$0.\dot{x}y\dot{z} \longrightarrow$ A triple recurring digit will be a fraction over 999 — $\frac{xyz}{999}$

More complex recurring decimals

Be in a position to eliminate the Decimal numbers

$0.20\dot{5}$

$(\times 100)\ 0.20\dot{5} = x\ (\times 100)$

Move recurring decimal up to the decimal point

$100x = 20.\dot{5}$

$(\times 10)\quad(\times 10)$

$1000x = 205.\dot{5}$

Be in a position to eliminate the Decimal numbers

$1000x = 205.\dot{5}$

$-\ 100x = 20.\dot{5}$

$900x = 185$

$x = \frac{185}{900} = \frac{37}{180}$

- Create equation by labelling the decimal x
- Move recurring decimal up to the decimal point by multiplying by 10/100 etc.
- Be in a position to eliminate the recurring decimal by multiplying again by 10/100 etc.
- Subtract two equations to eliminate recurring decimals and convert into fraction
- Simplify your answer

Algebra — Higher

Arithmetic sequence and Quadratic sequence

Finding the n^{th} term rule

Sequence	5	8	11	14
Times table	3	6	9	12
Extra bit	+2	+2	+2	+2

+3 +3 +3

$3n + 2$

- Find the common difference (this will be your n coefficient)
- Write times table underneath sequence (of your n coefficient)
- Sequence minus times table (this is your extra bit)

General formula

$n^{th}\ term = 1^{st}\ term + common\ difference \times (n-1)$

Has the form $an^2 + bn + c$. A second layer difference

5 9 15 23 33 ...

+4 +6 +8 +10

+2 +2 +2

Halve 2^{nd} layer difference for n^2 coefficient

$1n^2 + bn + c$ — Find linear sequence

Sequence	5	9	15	23
$1n^2$	1	4	9	16
Subtract	4	5	6	7

n^{th} term rule of this $= n + 3$

$1n^2 + 1n + 3$

Changing the subject

Often it is useful to re-arrange a formula to make a different variable the subject

Make l the subject of the formula

$P = 4l \Rightarrow \frac{P}{4} = l$

$\div 4 \quad \div 4$

Use inverse operations

$y = \frac{18t - 3}{p}$ — Make t the subject

$\times p \quad +3 \quad \div 18$

$t = \frac{py + 3}{18}$

Sometimes a variable will appear more than once in a formula

Make x the subject of the formula:

$a = 5x + xy \rightarrow a = x(5 + y)$

Factorise first — $\frac{a}{5 + y} = x$

Advanced Laws of Indices

Negative Indices

$x^{-n} \Rightarrow {}^{1}/_{x^n}$

- Find Reciprocal
- Apply Positive Power
- Apply top and bottom

$z^{-3} = \left(\frac{1}{z}\right)^3 = \frac{1}{z^3}$ $\qquad 6^{-2} = \left(\frac{1}{6}\right)^2 = \frac{1}{36}$

Fractional Indices

$x^{\frac{m}{n}} = (\sqrt[n]{x})^m$

- Root by denominator first
- Then power of numerator

$x^{\frac{1}{2}} = \sqrt{x}$ $\quad x^{\frac{1}{3}} = \sqrt[3]{x}$ $\quad x^{\frac{1}{4}} = \sqrt[4]{x}$ $\quad x^{\frac{2}{3}} = (\sqrt[3]{x})^2$

$64^{\frac{2}{3}} = (\sqrt[3]{64})^2 = (4)^2 = 16$

Negative Fractional Indices

$x^{-\frac{a}{b}} = \frac{1}{(\sqrt[b]{x})^a}$

Negative Fractional Powers: Apply reciprocal first!

$9^{-\frac{3}{2}} = \frac{1}{9^{\frac{3}{2}}} = \frac{1}{(\sqrt{9})^3} = \frac{1}{(3)^3}$

Quadratics

$ax^2 + bx + c \Rightarrow (x + p)^2 + q$

$x^2 + bx + c$

Halve the coefficient of b

$\left(x + {}^{b}/_{2}\right)^2 - \left({}^{b}/_{2}\right)^2 + c$

Simplify

$(x + p)^2 + q$

$x^2 + 6x - 2 \Rightarrow (x + 3)^2 - (3)^2 - 2$

$(x + 3)^2 - 11$

Solving equations by completing the square

Complete the square — Solve equation

$x^2 - 10x + 15 = 0 \Rightarrow (x - 5)^2 - 10 = 0$

$(x - 5)^2 = +10 \Rightarrow x - 5 = \pm\sqrt{10}$

$x = 5 \pm \sqrt{10} \Rightarrow x = 5 + \sqrt{10} = 8.16$, $x = 5 - \sqrt{10} = 1.84$

Completing the square $a \neq 1$

$3x^2 + 6x - 4 = 0$

Factor out the three

$3\left(x^2 + 2x - \frac{4}{3}\right) = 0$

Complete the square on this expression

$3\left((x + 1)^2 - (1)^2 - \frac{4}{3}\right) = 0$

Complete the square.

$3\left((x + 1)^2 - \frac{7}{3}\right) = 0$

Expand

$3(x + 1)^2 - 7 = 0$

Solve for x.

$3(x + 1)^2 - 7 = 0$ — Add 7 to both sides

$3(x + 1)^2 = 7$ — Divide both sides by 3

$(x + 1)^2 = \frac{7}{3}$ — Square root both sides

$x + 1 = \pm\sqrt{\frac{7}{3}}$ — Subtract 1 from both sides — $x = -1 \pm \sqrt{\frac{7}{3}}$ — $x = 0.528$ or -2.528

Functions

Think of it as a machine that has an input which is processed by the function to give an output.

$f(x) = x + 2$

Input → Function → Output

x → $+2$ → Output

$f(4) = 6 \quad f(-3) = -1$

Substitute input into the function to generate output value

Using functions

$g(x) = \sqrt{4x - 3}$ find $g(21)$

$g(21) = \sqrt{4x - 3}$ — Substitute input into the function and calculate

$g(21) = \sqrt{81} \Rightarrow g(21) = 9$

$f(t) = 3t^2 + 2$ find $f(2)$

$f(2) = 3t^2 + 2$ — Substitute input into the function and calculate

$f(2) = 12 + 2 \Rightarrow f(2) = 14$

Inverse functions

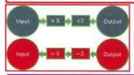
A function that performs the opposite process of the original function — Normal function $f(x)$

You have been given the value of the input. You have been given the output and need to work out the value of the input. — Inverse function $f^{-1}(x)$

Two ways to solve problems involving inverse functions

$f(x) = 5x + 2$

Function machine method



Subject of Formula method

$y = 5x + 2 \Rightarrow y - 2 = 5x$

$f^{-1}(x) = \frac{x - 2}{5} \Leftarrow \frac{y - 2}{5} = x$

Composite functions

The combination of two or more functions to create a new function

$f(x) = 2x + 2$ and $g(x) = x - 2$. Find $fg(x)$

The output of $g(x)$ will form the input of $f(x)$

$g(x) = x - 2$

$f(x) = 2x + 2$

$f(x - 2) = 2x + 2$

$f(x - 2) = 2(x - 2) + 2$

$fg(x) = 2x - 2$

Simultaneous equations

Equations involving two or more unknowns that are to have the same values in each equation

$4x + 3y = 5$, $3x + 2y = 4$ $\quad 2x - 3y = 4$, $5x + 2y = 1$ $\quad 3y + 10x = 7$, $y = 2x + 1$

Linear equations (Elimination method)

Multiply equations to get matching coefficients — Add/subtract equations — Substitute to find second variable

$4x + 3y = 5 \ \times 3$, $3x + 2y = 4 \ \times 4 \Rightarrow 12x + 9y = 15$, $-12x + 8y = 16$, $y = -1$

Substitute $y = -1$ into equation 2

$3x + 2(-1) = 4 \Rightarrow 3x - 2 = 4 \Rightarrow x = 2$

Matching coefficients: Same signs (Subtract the equations) Opposite signs (Add the equations)

Linear equations (Substitution method)

$3y + 10x = 7$, $y - 2x = 1 \Rightarrow 3y + 10x = 7$, $y = 2x + 1$ — Rearrange to get a single variable on its own

Substitute $y = 2x + 1$ into equation 1 — Substitute equations to find first variable

$3(2x + 1) + 10x = 7$

$16x + 3 = 7$

$x = 0.25$

Substitute $x = 0.25$ into equation 2 — Substitute to find second variable

$y = 2(0.25) + 1 \Rightarrow y = 1.5$

Quadratic equations (Substitution method)

$y = x + 6$, $y = x^2 - 2x + 2 \Rightarrow x + 6 = x^2 - 2x + 2$, $0 = x^2 - 3x - 4$

Substitute equation into quadratic and rearrange to = 0

$(x + 1)(x - 4) = 0 \qquad x = -1$ or $+4$

Factorise and find two solutions for variable

$y = (-1) + 6$, $y = 5 \qquad y = (4) + 6$, $y = 10$

Substitute each answer to find other pair of solutions

Proportion

Direct Proportion

y is directly proportional to x

$y \propto x$ — Constant of proportionality

$y = k \times x$ — k is the rate of change

Solving direct proportion problems

p is directly proportional to t.

$p = 24, t = 8$

a) Find p when $t = 7$

b) Find t when $p = 39$

Compare two values

$p = k \times t \Rightarrow 24 = k \times 8$

Work out the value of k

$24 = k \times 8 \Rightarrow \frac{24}{8} = k \Rightarrow 3 = k$

Form equation to solve problems

$p = 3 \times t \qquad$ a) $p = 3 \times 7 = 21$

b) $39 = 3 \times t \Rightarrow t = 13$

Inverse Proportion

y is inversely proportional to x

$y \propto \frac{1}{x} \qquad y = \frac{k}{x}$ — Constant of proportionality k

Solving inverse proportion problems

p is inversely proportional to t.

$p = 16, t = 2$

a) Find p when $t = 8$

b) Find t when $p = 64$

Compare two values

Work out the value of k

$p = \frac{k}{t} \qquad 16 = \frac{k}{2} \Rightarrow 32 = k$

Form equation to solve problems

$p = \frac{32}{t} \qquad$ a) $p = \frac{32}{8} = 4$

b) $64 = \frac{32}{t} \Rightarrow t = \frac{32}{64} = 0.5$

An equation where the highest power of the variable is 2

$ax^2 + bx + c$

Factorising $a = 1$ Quadratics

Aim: Convert quadratic into double brackets $\quad (x \pm \quad)(x \pm \quad)$

Sum and product rule — Establish Signs

$x^2 + bx + c$ — Add to make b — Multiply to make c

If c is positive — Signs are same: $x^2 + 5x + 6 \quad (x + 3)(x + 2)$

If c is negative — Signs are different: $x^2 + 5x - 6 \quad (x + 6)(x - 1)$

Example

$x^2 - 7x + 12 \qquad (x - \quad)(x - \quad)$

Positive c → Signs Same; Negative b → Both Minus

Factors of 12: 12×1, 6×2, 4×3 — Which pair make 7?

$(x - 4)(x - 3)$

Factorising $a \neq 1$ Quadratics

$(?x \pm \quad)(?x \pm \quad)$

Factors of a to find possible values

$5x^2 - 14x - 3 = (5x \pm \quad)(1x \pm \quad)$

$6x^2 + x - 2 = (3x \pm \quad)(2x \pm \quad)$ OR $(6x \pm \quad)(1x \pm \quad)$

Then find factors of c and see which satisfy b

Difference of Two Squares (DOTS)

$a^2 - b^2 = (a + b)(a - b)$

$x^2 - 81 = (x + 9)(x - 9)$

$4y^2 - 25 = (2y + 5)(2y - 5)$

Solving by factorising

$ax^2 + bx + c = 0 \Rightarrow (x \pm \quad)(x \pm \quad) = 0$

Factorise the quadratic – You may need to rearrange first

$x^2 + 8x + 7 = 0 \Rightarrow (x + 7)(x + 1) = 0$

$x = -7$ or -1

Find values for x that will make each bracket $= 0$

$2x^2 - 2x = 3(1 - x) \Rightarrow 2x^2 - 2x = 3 - 3x$

Expand and rearrange to $= 0$

$2x^2 + x - 3 = 0 \Rightarrow (2x + 3)(x - 1) = 0$

$x = -\frac{3}{2}$ or $+1$

The quadratic formula

The formula you need to know

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Substitute values into the formula to generate two answers for x

$5x^2 + 8x - 4$ — Identify values of a, b and c

$x = \frac{-(8) \pm \sqrt{(8)^2 - 4(5)(-4)}}{2(5)}$

Substitute and simplify

$x = \frac{-8 \pm \sqrt{144}}{10}$

Carry out two calculations

$x = 0.4$ or -2

Formulas I must memorise

The Quadratic Formula

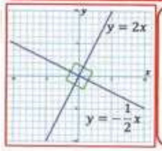
The solutions of $ax^2 + bx + c = 0$, where $a \neq 0$, are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Graphs — Higher

Perpendicular lines

Perpendicular lines are lines that **intersect** (cross) to **form 90° angles**



Gradients that are negative reciprocals of each other.

$$m_1 \times m_2 = -1$$

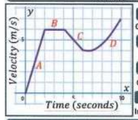
Invert = Reciprocal

Use gradients and coordinate points to calculate the equation of the line

Real life graphs and rate of change

Velocity – Time graphs

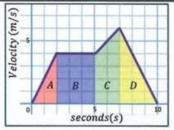
Velocity - Time graphs record the velocity of a particle moving along a straight line



- Stage A — Constant rate of acceleration
- Stage B — Steady 'constant' speed
- Stage C — Constant rate of deceleration
- Stage D — Increasing rate of acceleration

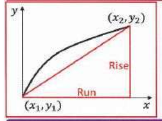
The gradient of the line is the acceleration.

Area under graph = Distance travelled



Average rate of change

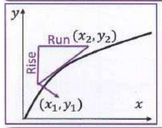
The rate of change **over a given interval**



- Create **chord** between two intervals
- Calculate gradient of chord
- Interpret gradient as a rate of change

Instantaneous rate of change

The rate of change at a **particular moment**



- Create **tangent** at specific point
- Calculate gradient of tangent
- Interpret gradient as a rate of change

Simultaneous equations on graphs

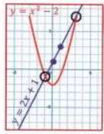
When we are given a system of equations, we can find the solutions to these equations algebraically (simultaneous equations) or **graphically.**

$2x + 3y = 18$, $y = x + 1$ — Solve simultaneously or **graph** each equation

$x = 3 \quad y = 4$

The point of **intersection** will be your **solutions**. The point where x and y have the same value for each equation.

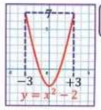
$y = x^2 - 2$, $y = 2x + 1$ — Solve simultaneously or **graph** each equation



Two intersections = two sets of solutions

$x = 3 \quad y = 7$
or
$x = -1 \quad y = -1$

Find the solutions of x when $x^2 - 2 = 7$



Plot graph and read off points

$x = 3$
or
$x = -3$

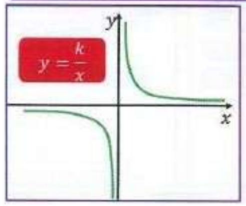
Plot each equation separately → Identify and read off the points of intersections for your solutions → Use graph to read off specific values for x and y

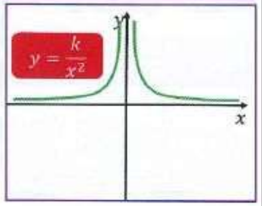
Proportion graphs

Inverse Proportion

$y \propto \frac{1}{x} \rightarrow y = \frac{k}{x}$ — As one value increases, the other decreases

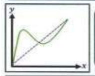
Inverse proportion will always give a **Curved** graph



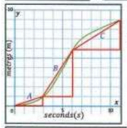


Gradients and areas

Estimating Gradients



Calculate average gradient from beginning to end. These are not very accurate and do not show the full picture

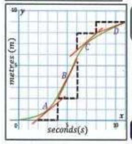


Break the graph down into smaller pieces to see what is happening

$Gradient\ A = 1/3 \Rightarrow 0.3m/s$

$Gradient\ B = 5/3 \Rightarrow 1.7\ m/s$

$Gradient\ C = 3/5 \Rightarrow 0.6m/s$



Find out what is happening at a particular point - **Tangents**

$Gradient\ A = \frac{1.5}{2}$ — 0.75m/s at 3 seconds

$Gradient\ B = \frac{3.5}{2}$ — 1.75m/s at 5 seconds

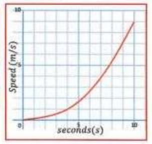
$Gradient\ C = \frac{1.5}{2}$ — 0.75m/s at 7 seconds

$Gradient\ D = \frac{1}{3}$ — 0.3m/s at 10 seconds

Estimate because tangents vary

Area under a curve

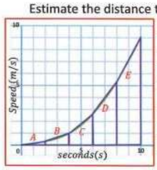
The area under a curve will enable you to estimate the total distance travelled in velocity time graphs



Formulae needed

- Trapezium: $\frac{1}{2}(a+b) \times h$
- Triangle: $\frac{1}{2}(b \times h)$

Estimate the distance travelled for the first ten seconds



$Area\ A = 0.5 \times 2 \times 0.5$ — $Area\ A = 0.5m$

$Area\ B = 0.5(0.5+1) \times 2$ — $Area\ B = 1.5m$

$Area\ C = 0.5(1+2.5) \times 2$ — $Area\ C = 3.5m$

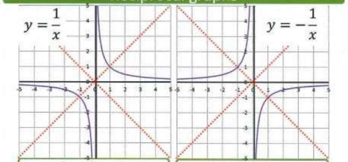
$Area\ D = 0.5(3+5) \times 2$ — $Area\ D = 8m$

$Area\ E = 0.5(5+9) \times 2$ — $Area\ E = 14m$

$27.5m$

Non linear graphs

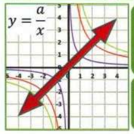
Reciprocal graphs



x-axis and y-axis are asymptotes

Two lines of symmetry

$y = \frac{1}{x}$, $y = \frac{4}{x}$, $y = \frac{6}{x}$

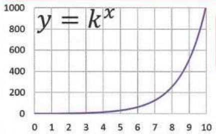


As a increases, the graphs move further away from the origin.

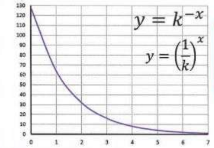
Points can be found by substitution

Exponential graphs

As x continues to increase, y continues to rise or fall at a continually faster or slower rate.

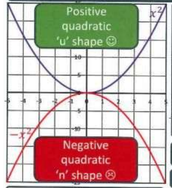


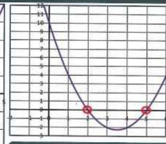
When $k > 1$, and x is positive, the graph will curve upwards



When $0 < k < 1$, or x is negative, the graph will curve downwards

Quadratic graphs





The points where the graph crosses the x axis are known as roots

These are the solutions to the quadratic when it equals zero

To plot graphs

Substitute the x values into equation to get y and plot points like coordinates

Maximum and minimum points

The line of symmetry runs through these points

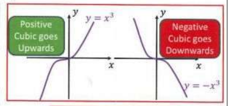
- Read off coordinate point from graph
- Find the midpoint of the roots and substitute into equation to calculate y –coordinate
- Complete the square in the form $(x-p)^2 + q$

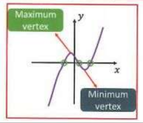
Max/min point can be found by ⇒ (p, q)

Cubic graphs

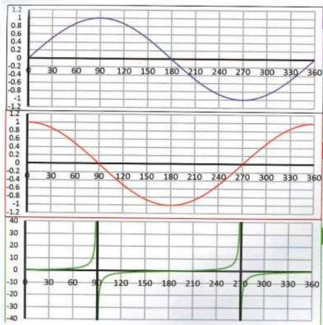
Can be defined as an equation where the highest power of the variable (usually x) is 3

$$y = x^3 + 3x^2 + 5x - 20$$





Trigonometric graphs



They are periodical — they will continue to repeat every 360° in both directions

There can be multiple answers for values of $Sin(\theta), Cos(\theta), Tan(\theta)$ — Use symmetry of graphs to solve problems

Sine — $Sin(\theta) = \frac{opp}{hyp}$

- x axis intercept at 0°, 180°, 360°
- Turning points at 90°, 270°

Cosine — $Cos(\theta) = \frac{adj}{hyp}$

- x axis intercept at 90°, 270°
- Turning points at 0°, 180°, 360°

Tangent — $Tan(\theta) = \frac{opp}{adj}$

- x axis intercept at 0°, 180°, 360°
- Asymptotes at 90°, 270°

Circles

There is a specific general formula for the equation of a circle

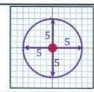
$$x^2 + y^2 = r^2$$

$x^2 + y^2 = r^2$ — Algebra / Number

- Centered at Origin
- Find the radius (r)

$x^2 + y^2 = 25$

Radius = 5



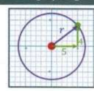
Finding the radius and equation using Pythagoras

$5^2 + 4^2 = r^2$

$41 = r^2$

$r = \sqrt{41}$

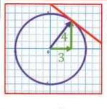
$x^2 + y^2 = 41$



We may be asked to find the equation of a tangent to a circle at a given point.

Gradient of Radius ⇒ Gradient of Tangent ⇒ Substitute in Point

What is the tangent to the circle $x^2 + y^2 = 25$ at the point (3,4)?



$\frac{Gradient}{(Radius)} = \frac{4}{3}$

$\frac{Gradient}{(Tangent)} = -\frac{3}{4}$

Circle Theorem: A radius always meets a tangent at a right-angle.

Perpendicular line

$y = -\frac{3}{4}x + c \Rightarrow$ Point (3,4) $\Rightarrow (4) = -\frac{3}{4}(3) + c \Rightarrow \frac{25}{4} = c$

$$y = -\frac{3}{4}x + \frac{25}{4}$$

Transformation of graphs

Translation

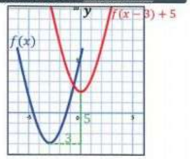
A translation can be defined as the movement 'sliding' of a shape to a new position

- $f(x) + a$ — The graph shifts up/down the $y-axis$ by a units
- $f(x + a)$ — The graph shifts left/right along the $x - axis$ by a units; $+ =$ Left, $- =$ Right

Adding to the function causes a **Translation**

Given $f(x)$, Sketch $f(x-3) + 5$

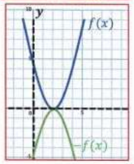
3 units right, 5 units up



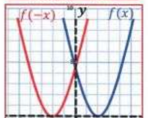
Reflection

Reflection: The replacement of each point on one side of a line by the point symmetrically placed on the other side of the line.

$y = -f(x)$ — The **outputs** are reversed. The graph is reflected in the $x - axis$



$y = f(-x)$ — The **inputs** are reversed. The graph is reflected in the $y - axis$



Geometry — Higher

Triangle properties

All three angles can be orientated to fit on a straight line → All angles in a triangle make 180°

$Straight\ line = 180°$

Calculate what you already know. $112° + 37° = 149°$

Subtract from 180°: $180° - 149°$, $x = 31°$

Exterior angle = Sum of two angles on opposite side.

$55° + 107° = x°$

$162° = x°$

Circles and sectors

$A = \pi r^2$ → Pi times the radius squared

Diameter is double the radius

$A = \pi \times 6.5^2$

$A = \pi \times 42.25$

$A = 132.73m^2$

Sector

$A = \frac{n°}{360}\pi r^2$ Calculate the proportion of the circle required then use area formula

$\frac{85°}{360°}(\pi \times 6^2)$

$26.7cm^2$

Circumference

$C = \pi d$

$C = 2\pi r$

The circumference is always about three time the length of the diameter

$C = \pi \times 12cm$

$C = 37.7cm$

Arc length

$L = \frac{n°}{360}\pi d$ Calculate the proportion of the circle required then use circumference formula

$\frac{85°}{360°}(\pi \times 12)$

$8.90cm$

Regular polygon properties

Knowledge of triangles is important

Number of sides	Number of △	Sum of interior angles	Regular interior angle	Regular exterior angle
3	1	180°	60°	120°
4	2	360°	90°	90°
5	3	540°	108°	72°
6	4	720°	120°	60°
7	5	900°	129°	51°
8	6	1080°	135°	45°
n	$(n-2)$	$(n-2) \times 180°$	$\frac{(n-2) \times 180°}{n}$	$360° \div n$

The number of triangles in a shape will always be **TWO** less than the number of sides

Hexagon

Octagon

Side, Side, Side — All three sides of one triangle are equal to the corresponding sides of the other triangle.

Side, Angle, Side — Two sides and the included angle are equal to the corresponding sides and included angle of the other triangle

Angle, Side, Angle — Two angles and one side of a triangle are equal to the corresponding angles and side of the other triangle

Right, Hypot, Side — Each triangle contains a right angle, the hypotenuses are equal in length as well as another equal comparative side.

Properties of 3D shapes

Pyramids

The volume of a pyramid is always $^1/_3$ of the prism that surrounds it

$$Volume = \frac{Area\ of\ base \times height}{3}$$

Calculate the volume of the square based pyramid if its height measures 15cm

$12cm^2 \times 15cm = 2160cm^3$

$2160cm^3 \div 3$

$720cm^3$

Curved area of cone

$Area = \pi rl$

Spheres

$$Volume = \frac{4}{3}\pi r^3$$

Find the volume of the sphere with diameter 16cm. Give your answer to 3 significant figures

$$Volume = \frac{4}{3} \times \pi \times (8)^3$$

$2144.660585cm^3$

$2140cm^3$

For a **hemisphere**, don't forget to **halve** your answer

Similar shape

Similar shapes are just **enlargements** of the original

The angles remain the same but the **lengths** have been **scaled up or down**

This **scale factor** needs to be calculated in order to solve problems involving similar shapes.

Find two comparative lengths.

$$Scale\ factor = \frac{New\ shape}{Original\ shape}$$

$Scale\ factor = \frac{12cm}{3cm} = 4$

Original Shape → x scale factor → New Shape; ÷ scale factor

Similarity in more than 1D

Area and Volume scale factors will need to be calculated

$Area\ scale\ factor = Scale\ factor^2$

$Volume\ scale\ factor = Scale\ factor^3$

Original Shape — x scale factor — ÷ scale factor — New Shape

Work out **all** scale factors first

Scale factor = 2

Area scale factor = $(2)^2 = 4$

Volume scale factor = $(2)^3 = 8$

Loci

The locus of points from a point is a circle

The locus of points between two lines is the angle bisector

The locus of points between two points is the perpendicular bisector

Notice how the locus of points around a corner is CURVED

Trigonometry

Trigonometry is used to calculate sides lengths and angles in triangles using three important ratios. Sine, Cosine and Tangent

The angle is often described as theta which is the Greek letter (θ)

Opposite, Hypotenuse, Adjacent, θ

The Sine rule formula

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ (Missing length) or $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ (Missing angle)

We tend to use the Sine rule if we know an **angle** and its **opposite length**

$\frac{35}{\sin(125)} = \frac{x}{\sin(35)}$ $\times \sin(35)$

$\frac{35}{\sin(125)} \times \sin(35) = x$ $\quad$ $24.51m$

$\frac{\sin(x°)}{28} = \frac{\sin(40)}{21}$ $\times 28$ $\quad$ $x = 59°$

$\sin(x) = \frac{\sin(40) \times 28}{21}$ $\quad$ $x° = \sin^{-1}\left(\frac{\sin(40) \times 28}{21}\right)$

The Cosine rule formula

$a^2 = b^2 + c^2 - 2bc\cos A$ → Missing length

$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ → Missing angle

If we know two sides **AND** the included angle

$x^2 = (90)^2 + (35)^2 - (2(90)(35)\cos(68°))$

$x^2 = 6964.978 \ldots$

$x = 83.46km$

$\cos x = \frac{(60)^2 + (35)^2 - (90)^2}{2(60)(90)}$

$\cos x = -0.303 \ldots$

$x = \cos^{-1}(-0.303 \ldots)$ $\quad$ $x = 107.7°$

$Area = {}^1/_2(7)(10)\sin(73°)$

$Area = {}^1/_2 \times 7 \times 10 \sin(73°)$

$Area = 33.47cm^2$

We have two sides and the included angle

Sine Function

Work out side lengths and angles when given the **opposite** and **hypotenuse**

$\sin\theta = \frac{opposite}{hypotenuse}$

$\sin\theta$	Value
0°	0
30°	0.5
45°	$\frac{\sqrt{2}}{2} = 0.707$
60°	$\frac{\sqrt{3}}{2} = 0.866$
90°	1

To get $\sin^{-1}$ you have to press shift first

Only used to calculate an angle

Cosine Function

Work out side lengths and angles when given the **adjacent** and **hypotenuse**

$\cos\theta = \frac{adjacent}{hypotenuse}$

$\cos\theta$	Value
0°	1
30°	$\frac{\sqrt{3}}{2} = 0.866$
45°	$\frac{\sqrt{2}}{2} = 0.707$
60°	0.5
90°	0

To get $\cos^{-1}$ you have to press shift first

Only used to calculate an angle

Tangent Function

Work out side lengths and angles when given the **adjacent** and **opposite**

$\tan\theta = \frac{opposite}{adjacent}$

$\tan\theta$	Value
0°	0
30°	$\frac{\sqrt{3}}{3}$
45°	1
60°	$\sqrt{3}$

To get $\tan^{-1}$ you have to press shift first

Only used to calculate an angle

When to use trigonometry

Find a missing side when given a length and an angle

or

Find an angle when given two lengths

Label the triangle accurately with O,A,H

Decide which ratio to apply

Use the triangles to help

Calculating a missing side

Cover up what you need to find

$x = 14 \div \cos(42°)$

$= 18.84$

Cover up what you need to find

$x = \tan(64°) \times 11.5$

$= 23.58$

Calculating a missing angle

Cover up what you need to find

x is an angle so use inverse

$x = \sin^{-1}(14 \div 37.8)$

$= 21.7°$

Alternate process

Use the functions and substitute

$\sin\theta = \frac{opposite}{hypotenuse}$ $\quad$ $\tan\theta = \frac{opposite}{adjacent}$ $\quad$ $\cos\theta = \frac{adjacent}{hypotenuse}$

Calculating area of a triangle using $^1/_2\,ab\sin C$

Two sides and the included angle

Label triangle

Substitute values in and calculate

$Area = {}^1/_2\,ab\sin C$

Circle Theorems

Angle at the centre is twice the angle at the circumference — Look for the 'Arrow' Shape!

Angle subtended at circumference by a semicircle is 90° — Opposite angle to the diameter!

Tangents and radii meet at 90°

Tangents from a point have equal length — Tangent just touches the circumference

Angles in the same segment are equal — Look for the 'Bow' Shape!

Opposite angles in a cyclic quadrilateral sum to 180° — $A + C = 180°$ $\quad$ $B + D = 180°$

Alternate Segment Theorem — Tangent

Bearing

Lines North are Parallel

Co-Interior Angles: $75° + \theta° = 180°$; $\theta° = 180° - 75° = 105°$

Angles around a point: $105° + \phi° = 360°$; $\phi° = 360° - 105° = 255°$; The bearing of A from B is 255°

Angle clockwise from North: 075°

Always 3 digits: $75° \rightarrow 075°$; $4° \rightarrow 004°$

Sentence Structure Important: The bearing of B from A is 075°

310° Clockwise

From A to B

Data Handling — Higher

Sampling

We collect and analyse data to give us information about a population

Census

Data is collected from the WHOLE population

Can take a very long time to collect the information

Sample

Data is collected from PART of the population

Quicker to collect the data and the data can be used to describe the whole population

Random

Your sample is randomly selected

Each member assigned a number
Numbers randomly generated
Those numbers used in sample

Stratified

Proportionate numbers from each group selected to make sample

$$\frac{\text{Amount in group}}{\text{Total number}} \times \text{Sample size}$$

Bias

Some situations can cause bias and make the sample unrepresentative

When and where the sample is taken?
Is the sample large enough?
Who is in the sample?

Histograms

A special type of bar chart for grouped data

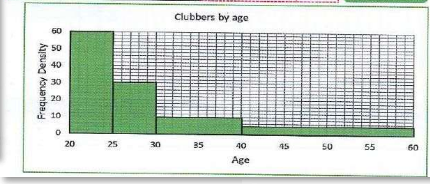
Frequency Density on the vertical axis

Bars can be different widths depending on the group size

$$\text{Frequency Density} = \frac{\text{Frequency}}{\text{Class Width}}$$

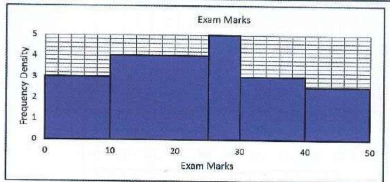
Age	Frequency		Class Width	F.D.
$20 \le x < 25$	300	÷	5	60
$25 \le x < 30$	150	÷	5	30
$30 \le x < 40$	100	÷	10	10
$40 \le x < 60$	100	÷	20	5

Create the Frequency density column



Calculating Frequency from Histograms

$$\text{Frequency} = F.D. \times \text{Class Width}$$



Age	F.D.		Class Width	Frequency
$0 < x \le 10$	3	×	10	30
$10 < x \le 25$	4	×	15	60
$25 < x \le 30$	5	×	5	25
$30 < x \le 40$	3	×	10	30
$40 < x \le 50$	2.5	×	10	25

Cumulative frequency table and graph

Cumulative Frequency tables

Running total (add up as we go down)

Time	Frequency	Cumulative Frequency
$45 \le x < 48$	3	3
$48 \le x < 50$	8	11
$50 \le x < 52$	16	27
$52 \le x < 55$	12	39
$55 \le x < 60$	11	50
$60 \le x < 70$	2	52

Add, Equals

The difference between cumulative frequency values will tell you the frequency

May be asked to calculate percentages

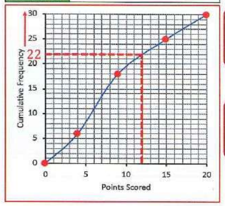
$$\% = \frac{\text{Amount}}{\text{Total}} \times 100$$

Cumulative Frequency graphs

Points	Cumulative Frequency
0 – 4	6
5 – 9	18
10 – 15	25
16 – 20	30

Always plot each point at the end of the group

Draw a smooth line through the points



How many games did the player score more than 12 points?

$30 - 22 = 8$ games

What percentage of games does the player score less than 12 points?

$\frac{22}{30} \times 100 = 73\%$

Averages from grouped frequency table

Weight	Frequency		Mid-point		fx
$0kg \le x \le 2kg$	12	×	1	→	12
$2kg < x \le 4kg$	3	×	3	→	9
$4kg < x \le 6kg$	9	×	5	→	45
$6kg < x \le 8kg$	10	×	7	→	70
Sum of Freq	34		Sum of fx		136

Two extra columns are created, the midpoint of each group and the fx column

This enables us to find out an estimate for the mean

Mean

$$\text{Mean} = \frac{\text{Total of all values}}{\text{number of values}}$$

$$\text{Mean} = \frac{\text{Total of } fx \text{ column}}{\text{Total frequency}}$$

$$\text{Mean} = \frac{136}{34} = 4kg$$

Median

Median = Middle value (Numbers written in order)

$$Location = \frac{n+1}{2}$$

$$Location = \frac{34+1}{2} = 17.5$$

The median lies between 17th and 18th value

Weight	Frequency	
$0kg \le x \le 2kg$	12	12
$2kg < x \le 4kg$	3	15
$4kg < x \le 6kg$	9	22
$6kg < x \le 8kg$	10	
Sum of Freq	34	

Add down the frequency column. When location value has been exceeded, that is the group where the median lies.

Median class = $4kg < x \le 6kg$

Mode

Mode = Most common value/item

The category with the highest frequency

$0kg \le x \le 2kg$

Range

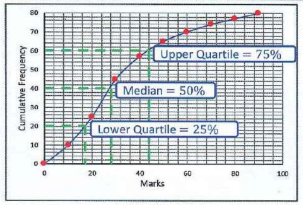
Range = Largest - Smallest

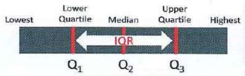
$8 - 0 = 8kg$

Quartiles and Box plots

Quartiles

Interested in specific points along the distribution





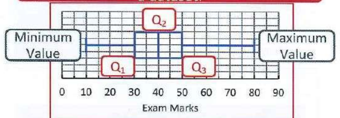
IQR = $Q_3 - Q_1$

The IQR provides a measure of the spread of the middle 50% of the data.

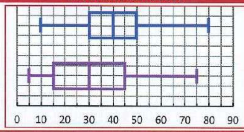
It is less affected by outliers than the range.

Box plots

Box plots allow us to visualise the spread of a dataset.



Also used to compare two or more different datasets



Look to compare medians, IQR and range

Venn diagrams

Venn diagrams

A set is a collection of things, called elements

The set of prime numbers less than 12: $A = \{2, 3, 5, 7, 11\}$

Intersections ($\cap$) and Unions ($\cup$)

$A = \{2, 3, 5, 7, 11\}$ $B = \{1, 3, 5, 7, 9\}$

$A \cap B = \{3, 5, 7\}$ – Intersection of A and B

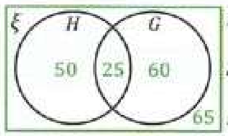
The crossover between sets

$A \cup B = \{1, 2, 3, 5, 7, 9, 11\}$ – Union of A and B

All values in the sets

$A' \cap B = \{1, 9\}$

Values in B but not in A



$P(G) = \frac{85}{200}$

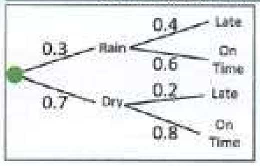
$P(H \cap G) = \frac{25}{200}$

$P(H \cap G') = \frac{50}{200}$

Probability trees

Dependent events

Probability trees where the outcome of one events affects the outcome of the next event e.g. no replacement, weather etc.



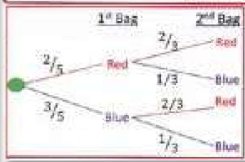
$P(\text{Rain and late}) = (0.3 \times 0.4) = 0.12$

$P(\text{On time}) = (0.18 + 0.56) = 0.74$

When dealing with no replacement, remember to reduce the denominator by one for the second event

Probability trees

Probability trees are really useful to calculate the probabilities of combined events happening



Multiply along branches

$P(\text{Red and Red}) = \frac{4}{15}$

$P(1 \text{ Red and } 1 \text{ Blue}) = \frac{2}{15} + \frac{6}{15} = \frac{8}{15}$

Add together all combinations

Probability

Where $P(A)$ is the probability of outcome A and $P(B)$ is the probability of outcome B:

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

Conditional Probability

$P(A \text{ and } B) = P(A \text{ given } B) \times P(B)$

Formulas I must memorise